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MOST HUMBLY DEDICATED TO HIS MAJESTY,

BY HIS MAJESTY'S MOST GRACIOUS PERMISSION,

A

TREATISE

ON

TOPOGRAPHY:

IN WHICH

THE SCIENCE AND PRACTICAL DETAIL

OF

Trigonometrical Surveying

ARE EXPLAINED:

Together with their Application to

SURVEYING IN GENERAL;

BESIDES VARIOUS OTHER SUBJECTS,

Including the Description and Use of the *Repeating Circle* and BORDA'S *Reflecting Circle;* as well as Tables from which the Height of any Place may be ascertained with the Barometer, by the mere Subtraction of Two Numbers:

AND ALSO

A Translation of the celebrated Essay on *Military Reconnoitring, Descriptive* and *Military Memoirs*, written by COLONEL ALLENT, of the French Royal Engineers.

COMPILED AND PARTLY WRITTEN

BY

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N. B. The Specimen of Calculation alluded to in page 106, follows immediately Table V.

INTRODUCTION.

THE great importance of Geography and Topography, both in a civil and military point of view, induced the French government, a few years since, to publish, at its own expence, a work entitled *Memorial Topographique et Militaire,* which is now well known in England. The materials of that work were selected and arranged under the direction of the *Dépôt Général de la Guerre,** by men most profoundly skilled in the subjects which it contains, and its publication has been highly beneficial to the French; first, it has afforded them a desirable uniformity in the execution of Topography, and therefore prevented the

* The *Dépôt Général de la Guerre* in France, is a government institution, established in 1688, in the reign of Lewis the Fourteenth, under the direction of which are maps, plans, reconnoissances, military memoirs, &c.

adoption of such arbitrary methods as would not always lead to results equally satisfactory. Secondly, this single work, both the extent and price of which are moderate, compared with the variety of subjects which it comprehends, now offers to the French a valuable collection of precepts which were previously dispersed through several very voluminous and expensive publications.—Lastly, it relieves the young French Topographer from the anxiety to which he was before subject, respecting the method he ought to adopt, as it is now sufficient for him to follow the path which men of consummate knowledge and experience have marked out for him in the *Memorial*.

The French *Memorial Topographique et Militaire* forms the basis of the *Treatise on Topography,* which is now offered to English readers in their own language; and, as it is to be regretted that the limits of that work did not admit of less conciseness in the explanations relative to practical subjects, this kind of deficiency, in the French publication, has been fully supplied in the second volume of the English treatise; at least, as far as concerns the detail of Geodesic Operations. With regard to the application of Theory to Trigonometrical Surveying, the reader can undoubtedly not have a better guide than Colonel Mudge's account of the *Trigonometrical Survey of England and Wales,* carried on under the direction of that Officer.

This treatise is divided into *four* parts, illustrated by a variety of suitable plates; the *first part* contains all the Geodesic Operations explained in the *Memorial Topographique et Militaire*, with appropriate tables for the principal requisite reductions; most of the formulæ in this part have been selected from the works of Delambre, Laplace, Legendre, and Puissant, and these formulæ, as well as the numerical examples by which they are illustrated, have been reduced to English measures. The same part includes, likewise, a minute description, accompanied with figures, of the *Repeating Circle* and the method of using it; as well as the description and use of *Borda's Reflecting Circle*.

Part the Second consists of a translation of *Biot's* Essay on finding the altitudes of places by means of the barometer; including a brief historical account of the barometrical formula, and its complete demonstration by the simple elements of algebra; as well as tables, from which the difference of altitude of any two places may be obtained by the simple subtraction of two numbers; in this part, the linear measures have been reduced from French *metres* to English *yards*, and the heights of the thermometer from degrees of the *centesimal scale* to those of *Fahrenheit's*, generally used in this country. These reductions have necessarily caused the principal table to be recalculated from the reduced formula, and extended from forty-two columns to seventy-four; with some additional lines in

length, on account of the number of yards exceeding that of metres in a given space; in the first half of the table, where it was most essential, the numbers have also been carried to one more place of decimals in this treatise than in the French essay.

Part the Third comprehends, besides observations relative to the triangles of the second order, the method of forming triangles on the ground, and of resolving them by plane-trigonometry, in a manner analogous to that which is used in trigonometrical surveys. Important remarks and select problems follow this explanation, which immediately precedes a description of the implements used for surveying, and an illustration of the methods of making scales, tracing a right line on the ground between two objects, and measuring it with a chain or with rods; *minor surveys,* performed *with* or *without* calculation, are then considered, as well as the methods of transferring to paper, in both cases, the points of the ground which have been taken. A general description of the plane-table is next given, and the use of this instrument explained in the various circumstances in which it may be employed. The surveying compass is also described, and its use illustrated by practical examples. Lastly, some easy methods of levelling, and of taking a profile of a ground, will be found in this part of the treatise.

Part the Fourth contains the translation of an

Essay on Military Reconnoitring, written by *Allent,* a lieutenant colonel in the French engineers, with the notes and plates annexed to it. The excellence of this essay, which the *Depot General de la Guerre* has published in the *Memorial Topographique et Militaire,* is so generally acknowledged, as to render it unnecessary to dwell on its merits in this place; it shall only be observed, that the Author's extensive military knowledge has enabled him to view military reconnoitring in its true light; wherefore, instead of confining himself within the narrow bonds of former authors, he has embraced all the subjects essentially connected with this very important branch of military service.—He first points out the various kinds of information which war requires, and then investigates the means of procuring them. This leads him to the subject of military reconnoitring; and, after noticing the instructions which officers engaged in that operation may receive, he specifies the materials which will serve them in forming the outlines of their work. He next considers the instruments and processes by which a sufficient approximation will be obtained in military reconnoitring, with respect to the fundamental triangles, the particular determination of a place, and the heights and distances of objects. Surveys in which distances are measured with rods, &c. or only by pacing, or such as are carried on by the eye alone, and without any actual measurement, are likewise the

subjects of his explanations, together with other operations that may be performed without instruments.

Lastly, he gives proper directions for drawing maps adapted to military reconnoitring, and for the formation of descriptive and military memoirs.

CONTENTS

OF

VOL. I.

PART THE FIRST.

CHAPTER I.

On Geodesic Operations, the Description and Use of the Repeating Circle, the Description and Use of BORDA's *Reflecting Circle, and the Theory of Spirit Levels.*

CHAPTER II.

TABLES

RELATIVE TO PART THE FIRST.

PART THE SECOND.

Portable Barometrical Tables, giving the Difference of Level by a Simple Subtraction; with an Introduction containing the History of the Barometrical Formula, and its Complete Demonstration by the Simple Elements of Algebra, By M. BIOT. *Translated and adapted to English Instruments and Divisions.*

TABLES

RELATIVE TO PART THE SECOND.

A TREATISE ON TOPOGRAPHY.

PART THE FIRST.

On Geodesic Operations, the Description and Use of the repeating Circle, the Description and Use of Borda's Reflecting Circle, and the Theory of Spirit Levels.

GEODESIA properly signifies *measure of the ground;* the signification of this word, however, has been extended to those operations, the object of which is to determine the positions and respective distances of the different points on the surface of the globe. Thus the operations which serve to determine the length of terrestrial degrees, and those by which the outlines of geographical maps are established, are *geodesic* operations.

The process of these operations, the description and use of instruments employed in them, and the formulæ used in the resulting calculations, are now to be explained. Each formula will be accompanied by a numerical example, and the necessary tables for simplifying the different reductions will be subjoined.

Most of the formulæ here used have been extracted from Delambre's work, *On the Determination of an Arc of the Meridian.* The Memoirs of the Academy of Sciences, for 1778 and 1787, the measure of degrees in Peru, by Bouguer and La Condamine; the astronomical travels of P. Boscovich; the operations in France for

ascertaining the difference of longitude between the observatories of Greenwich and Paris, by Cassini; the Connoissance des Temps, year 6, (these last two works contain a description of the repeating circle,) Lalande's Astronomy; Cagnoli's Trigonometry; and the description and use of the reflecting circle, by Borda, may also be consulted.

Summary Explanations of the Operations by which the general outlines of large Maps are established.

The general outline is composed of triangles of the first order, formed by the longest lines which the localities of the situation will permit, the telescopes, however, magnifying sufficiently to be directed to the object with precision. All these triangles ought to be connected together, and to form, as much as possible, a kind of continued net-work in all directions. The choice of triangles is not indifferent; it is known that equilateral triangles are those in which small errors committed in observing the angles have the least influence on the length of the sides: but it would be very difficult to form a series of triangles by admitting those only that are nearly equilateral; and the invention of the repeating circle, with which angles may be observed within one second of the truth, gives a greater latitude in the choice of triangles. In general, however, all angles below 30° and above 120°, ought to be avoided.

The repeating circle is the only instrument which should be employed at present for measuring the angles of triangles of the first order,* if it be thirty-five centi-

* The author appears to have overlooked the excellent Theodolite with which the conductors of the Trigonometrical survey in this country have taken their angles with so much accuracy.

metres, in diameter, or nearly 14 English inches, it will be sufficient for geodesic operations; if greater, it would be troublesome to convey from place to place, and could not be placed in many situations where it is necessary to use it; if less, it would not afford sufficient accuracy, and the telescopes would not have sufficient power.

This instrument, which is brought into the plane of the objects with great facility, serves also to take the apparent zenith distances; that is to say, for determining the elements of the reduction of an angle to the horizon, and for finding the difference of level of the stations.

If the formation of the country in which the operations are carried on does not present objects conveniently situated for forming a suitable chain of triangles, this inconvenience is to be remedied by signals erected at elevated places. These signals are generally preferable to steeples or towers, which are frequently used for directing the sights of the instrument, because that form may be given to them which leaves the least uncertainty relative to the points to which the sights were directed. Experience has shown that the vertices of quadrangular pyramids, and balls of about a foot and a half diameter, are objects which leave the least uncertainty on this head.

We can seldom place ourselves exactly at the mathematical point chosen for the vertex of the angle which is to be observed, and when we cannot, this angle wants a correction, known by the name of *reduction to the centre*. The elements ought to be determined with great precision; for it may happen that an error of one or two seconds may be committed in the reduction.

The place where the observation is made is commonly not the point *de mire*, or that to which the sights of the instrument were directed; it is therefore

necessary to measure the distance of the place of observation from this point, in order to reduce the observed zenith distances to the summit of the object.

Frequently, when the sun shines on the object, we can only see a *phase;* and the apparent middle, that of the illuminated part, does not correspond with the axis of the object; then the observed angle wants a correction, which depends on the form of the object and the position of the sun.

We shall here recapitulate, in their natural order, the corrections which an angle observed with the repeating circle may require.

Correction for the eccentricity of the inferior telescope.

Correction for the phase of the signal when necessary.

Reduction to the centre.

Reduction to the horizontal plane by the knowledge of the apparent zenith distances of the two objects to which the sights have been directed: these distances having been previously reduced to the summit of the station, there remains only to correct the angles, for the spherical excess depending upon the surface of the triangles to which they appertain.

The measure of angles gives only the ratio of the length, or the *relative* length, of the sides of the triangles. In order to have their *absolute* lengths, or these expressed in known measures, it is necessary actually to measure one of them; and this is what is called the *measuring of a base.* This operation requires the most scrupulous attention; for its accuracy ought to equal that of the instrument employed in measuring the angles.

The actual distances between all the points chosen for the vertices of the triangles may therefore be known;

but terrestrial operations alone cannot give more. It is necessary to have recourse to astronomical observations to obtain the latitudes of the vertices of these triangles and the directions of their sides.

The latitude of one of the stations, situated nearly in the centre of the region, being observed, and also the azimuth, or inclination of one of the sides of the series from the meridian of the place, the azimuths of all the sides of the triangles may be calculated with these data, as well as the latitude of their vertices and their difference of longitude.

We may also calculate, by means of the latitude and azimuth, the distances of the vertices of the triangles from the meridian, and from a line perpendicular to it at the assumed origin of operation.

From these distances we may proceed to the latitudes and longitudes; or from these latter conclude the former; but it is at least as simple to determine the latitudes and longitudes independent of the distances from the meridian and its perpendicular.

As errors may happen to be committed either in measuring the angles, or in calculating successively the azimuths, latitudes, and longitudes, latitudes and azimuths of verification ought to be taken at other places, especially at the extremities of the chain, and which ought to agree, or very nearly, with those previously obtained by calculation. They should agree exactly, if the figure of the earth and its inequalities were well known. It is with the same view that several bases of verification, connected by the sides of the triangles, are measured. If all these measures agree with each other it is a proof of their accuracy.

The same operations also serve for determining the lengths of arcs of the meridian, and consequently fur-

nish valuable data relative to the figure and dimensions of the earth.

The sides of the triangles of the first order serve as bases to those of the second; for which a better instrument cannot be employed than the repeating circle; but it is not necessary that it should be so large as those used for the principal triangles; a diameter of about eight or ten inches will be sufficient, and the instrument will be very portable.

For triangles of the second order it will not be necessary to observe the angles with the accuracy of one or two seconds, as in the principal triangles; a precision of $5''$ or $6''$ will be sufficient, as the objects are much less distant from each other. The corrections to be applied to these angles will also be less numerous, as it will be sufficient to determine the correction for the eccentricity of the inferior telescope, and for the reduction to the centre of the station. The reduction to the horizontal plane may generally be dispensed with, at least when the zenith distances are not very considerable; for these reductions do not generally exceed a fraction of a second.

With these reduced angles, the sides of the triangles are to be calculated, after having divided the error equally among the three angles of each triangle, so that the sum may be equal to 180°. The distances of the vertices of the secondary triangles from the meridian and its perpendicular, may then be calculated, if necessary; or their latitudes and longitudes immediately determined. Then, by means of the distances, or of these latitudes and longitudes, the vertices of all the triangles may be laid down on paper, and those parts of the the ground transferred to the same plan, which were taken with the plane-table. If the latitudes and longitudes

are to be used for this purpose, the first step is to trace the meridians and parallels of latitude on the map; but should it be intended to use the distances from the meridian and its perpendicular, this meridian and perpendicular, with their parallels, must be drawn.

Description and Use of the Repeating Circle.

THE description and use of the repeating circle, such as given in the *Memorial Topographique et Militaire*, not appearing so perspicuous as could be wished, on account of the brevity of the explanation and the want of a figure, it has been thought proper to extract the following illustration of this subject from the second edition of Biot's *Astronomie Physique*.

The repeating circle may be substituted alone for the mural quadrant and transit instrument, which it replaces with great accuracy; it requires very small corrections, and may be conveniently transported from place to place, on account of its moderate size; finally, its great utility is not limited to astronomy only, but extends to geodesic operations, topographical detail, and to an infinity of physical researches of all kinds, into which it carries an unhoped-for precision.

The principal part of this instrument consists in a circular and vertical limb Z A P *fig.* 2. which may be turned about the vertical line C P drawn through its centre, and also about a horizontal axis, passing through the same centre. A telescope O C L, furnished with a micrometer and a vernier, turns about the centre C and may be successively passed over all points of the limb. The whole instrument is represented by *fig.* 1. And for an explanation of its use, let us return to *fig.* 2.

Let S be a distant and fixed object, the zenith distance of which is to be measured; or let us suppose S to be a star. For though the heavenly bodies change their places every instant on account of the diurnal motion, the effects of this change, during the interval of the observations, may be calculated and taken into the account; which brings the question to the same state as if we had only to observe fixed points. This being premised, the operation is as follows:—First, the vernier of the telescope must be fixed at the point zero of the division; the limb is then to be brought into the vertical plane of the heavenly body, by means of the azimuthal motion, and turned vertically about its centre, until the point S answers to the centre of the wires; this is represented by *fig.* 2. Now conceive a plumb line C P to pass through the centre of the limb; its direction produced will determine the zenith Z; and the arc A Z read upon the limb, will be the zenith distance. But the use of this line and the reading of the arc may be avoided in the following manner:—

Things being disposed as above described, an azimuthal motion is to be given to the limb about the vertical line which passes through its centre, until it has made half a revolution, and is thus brought again into the vertical plane of the heavenly body, see *fig.* 3. In this movement, the point Z of the limb has not changed, only, if the limb at first faced the east, it now faces the west, and, as the telescope is fixed to the limb, it follows that its actual direction L A C makes still the same angle with the vertical line; this telescope being loosened and directed to the heavenly body by making it revolve on the limb, its new direction C A′ S will answer to another point upon the limb, as A′; and, since the heavenly body is supposed stationary, the arc A′ Z will be exactly equal

to the arc A Z or the zenith distance. The total arc A A′, which the telescope has passed over, is therefore the double of that distance. Thus, by reading this arc, which is indicated by the vernier upon the division of the limb, and taking its half, we shall have the zenith distance without it being at all necessary to know the point Z, and consequently, without having any occasion for the plumb line.

This supposes, indeed, that the limb, in passing from the first observation to the second, has turned precisely about the vertical line, so that each of its points remains exactly at the same height above the horizontal plane. In order to ascertain this, a very sensible spirit level is fixed to the back of the limb, and parallel to its plane, which must be rendered horizontal in the first position of the circle, in which the limb is supposed to face the east,* then if the point Z of the limb is a little displaced by the motion when the limb is brought to face the west, it will be seen by the level which is displaced with it, as the bubble will not correspond any longer to the same points of the division of the tube. In this case the limb is to be brought to such a position as will bring the bubble within its former limits; thus, without knowing the point Z, we are assured that it has returned

* For this purpose there is a division drawn by the artist upon the tube of the level, or parallel to its length. The zero of this division is placed at the middle of the tube. For rectifying the level, its inclination is made to vary until the extremities of the bubble answer to divisions equally distant from the point zero. Rigorously, in observations with the circle, it is sufficient to place the bubble between the same physical points of the tube, in each pair of observations; and it is not at all necessary that its middle should correspond to the zero of the division.

to the same vertical plane. There are, in all circles, repelling screws by which the limb is made to move by insensible degrees, and the level rectified.

After a double zenith distance is thus found, a quadruple distance may be obtained in the following manner: without touching the telescope, the instrument is to be returned, and the limb brought to face the east, as it did at first. The telescope then takes the direction of C A′ L, *fig*. 4. If it were brought back towards the heavenly body, the limb remaining fixed, it would arrive at the point A, from which it was first moved, and the arc which it had passed over would be destroyed. Instead of this, it is left fixed in A′, but the limb must be turned vertically about its centre, until the heavenly body re-appears in the telescope at the centre of the wires. Then the point A′ is directed towards this body, and the point of departure descends to A, *fig*. 5. This being done, we find ourselves exactly in the same circumstances as at the commencement of the first operation, *fig*. 2: except that the point of departure is A′, that is to say, the end of the first arc passed over.* By setting off from this point, and operating the same as before, we may make a new double observation, which will bring the telescope to A″; and as the arc A′ A″ will be equal to A A′, the total arc A A′ A″ will be the quadruple of the zenith distance; by dividing it by four we shall have the single distance.

Having the quadruple distance, the sextuple may be

* By thus moving the limb in the third observation, the level which is attached to it and follows its motion ceases to be horizontal; but the observer, after loosening it, renders it again horizontal, and fixes it in that position by its repelling screws; the same must be done after each even observation.

obtained by the same process: the limb must be turned to the east, into its first position, and the point A″ brought towards the heavenly body, without moving the telescope. Then the new point of departure will be A″; a new double observation will bring the telescope to A‴, and the arc A A′ A″ A‴ will be the sextuple arc. By dividing it by six we shall obtain the single arc.

By continuing thus indefinitely, we shall obtain any multiple of the zenith distance that we wish; and, if we divide the total arc by the number of observations, we shall have the simple zenith distance. In this manner, the telescope may pass over several entire circumferences, the number of which it will be proper to estimate. But in order to avoid the trouble of reckoning them one by one, it is sufficient to read the double arc once, so as to know the simple distance; and when the observations are ended, it is easily seen what number of entire circumferences must be added, so that the total arc, divided by the number of observations, may again give the simple distance, determined approximately by the first reading.

Let us now examine in what the advantage of this multiplication consists. There would not be any, if the divisions marked upon the circle were mathematically exact, and if the pointing of the observer was always perfectly true; for then a single observation would give the zenith distance accurately. But as these conditions are impossible to be fulfilled in practice, the repetition of the angles supplies them by compensations.

First, as to the error of the divisions, we see that the arcs measured follow each other without interruption upon the limb, so that the point of the limb, which is the end of one observation, becomes the beginning of the next. Thus the sum of the observations, that is to say, the total arc passed over, will not contain any interme-

diate error, but solely the two errors of the extreme readings. Now these errors are diminished, as the indices of the circle carry four verniers, which are read separately, and the mean of which indicates the commencement and end of the total arc with a great probability of precision. Finally, the small error, which may still remain in the extreme readings, notwithstanding these precautions, extends to this total arc, which, as the reader has seen, is to be divided by the number of observations; wherefore the error, which is attenuated by this division, scarcely affects the simple arc passed over in a single observation; at least, if we suppose a sufficient number of observations. The errors of the divisions are therefore decreased in the repeating circle by the repetition itself, and the compensation which takes place between them is not only probable, but certain.

In order to understand how far this compensation may be carried, it is necessary to know that in our circles, which are about sixteen inches in diameter, the error of the divisions can not certainly amount to fifteen sexagesimal seconds. It will therefore be reduced to half a second at most, after thirty observations; what does it become then after eighty or one hundred? What does it become if, as may be done, and as frequently has been done, the series of observations for several days is continued upon the limb, without interruption, so that the two errors of the extreme readings only are divided upon a total arc which contains several thousand times the simple arc?

The error of the divisions is therefore as nothing in observations made with the circle. It is impossible that it should be so rigorously destroyed in the largest instruments, if they are not repeaters. The dexterity of the artist can never equal a mathematical process.

But there are other errors, which, in the use of the circle, are destroyed on the principle of probabilities, and which remain in other instruments. Such are the errors of the level, which, already very small in the first circles that were constructed, are still less in our actual circles, in which the level gives immediately the fractions of a second. Such are also the errors of the *pointing*, which not only prove very small, when the observations are carefully made, but are destroyed, like those of the level, by their fortuitous compensations in several thousand observations. Besides, large instruments, such as the mural quadrant for instance, are equally subject to these errors; for the use of this instrument does not remove the possibility of an error in directing the sight, and the plumb line points out the error of the level. But here, the small number of observations does not permit us to hope for a compensation so exact as with the circle. If we suppose the accuracy of the mean results to have a ratio compounded of the number of observations, and of the length of the radius of the instrument, a hundred observations, made with a circle of eight inches radius, would be equivalent to a single observation made with a mural quadrant of sixty-six feet radius. Where can similar instruments be found, and especially how can they be used for observations which require travelling?

After having explained, in general, the mechanism of the repetition and its advantages, it is necessary to enter into some details upon the particular verifications which the repeating circle requires before being used for observations.

First, the limb ought to be exactly vertical, and to remain in that position during the observation; had it deviated from it, the observer must rectify it. For this

purpose, a small spirit level is placed behind the limb, perpendicularly to its plane, which is attached to the horizontal axis about which the circle turns, see *fig.* 6. Thus the limb is vertical when the level is horizontal. Now, there is in repeating circles a repelling screw, by which the limb may be moved and rendered vertical, when the level shows that it has departed from that position.

With respect to this level, it is horizontal when the extremities of the air bubble which it contains terminate at two fixed lines drawn by the artist for that purpose. But it is useful to know how, in case of need, we may supply this requisite.

In fact, even supposing that the artist has perfectly regulated this level, it may happen to be deranged in its mounting, and will thus cease to be perpendicular to the limb; for which reason the observer must verify it, before he commences the observations: to do this, he fixes upon the limb two pincers P and Q, on which two extremely fine points are drawn, and placed at equal distances from the plane of the limb, to which the pincers are applied. Artists have very simple and accurate means of fulfilling this condition. At one of these points, the most elevated, the observer suspends a plumb line, and moves the limb until this line strike exactly against the other point. Then the limb is vertical, since, by the construction, the vertical line, drawn through the two points P and Q, is parallel to its plane. This being done, the observer uses the repelling screws of the small level in order to make it perfectly horizontal, and its variations will indicate afterwards whether the limb has departed from its vertical position. Nay, the sensibility of the level renders the use of it preferable for this purpose to that of the plumb line and particularly as it is much more convenient.

If it be wished also to verify the pincers themselves, nothing will be more easy. The small level being rectified, that is to say, rendered horizontal, turn the limb vertically, so that the pincer which was at the top comes to the bottom, and that the other which was at the bottom comes to the top. Then, suspend the plumb line afresh; and if it strike exactly as before upon the points P and Q, the limb is vertical, and the pincers are well regulated. In the contrary case, the deviation of the plumb line will be double of the error of parallellism. For let, for example, L L, *fig.* 8. be the direction of the limb in its first position, the vertical line P Q making with it a certain angle by the error of the pincers; in the turning of the limb, each of the points P and Q describes the circumference of a circle about the axis of rotation A C C′, which is perpendicular to the limb. The line P Q therefore describes, about this axis, a conical surface, of which the angle at the centre is P C′ P′, or Q′ C′ Q, P′ and Q′ denoting the new positions of the points P and Q after the limb has been turned; if from the point C′, C′ L′ be drawn parallel to the limb, this line, which remains fixed during the rotation, will divide the angle Q′ C′ P into two halves, each of which will be equal to the angle L′ C′ P, formed by the line P Q and the limb. Now, when we suspend afresh the plumb line at Q′, the quantity Q″ Q′ P′, by which it deviates from the line P′ Q′, will be equal to the angle at the centre Q′ C′ P: this deviation will consequently be double the error of parallelism.

If an error of this kind should be perceived, it must be corrected; for this purpose, there should be, upon the pincers, repelling screws, by which the points P and Q may be moved as required. These points may therefore be moved so as to destroy half the observed devia-

tion, and with the pincers, thus regulated, the limb is restored to its vertical position. But as it is very difficult to make this bisection exactly the first time, when we have ascertained the vertical position of the limb very nearly, by means of the rectified pincers, we avail ourselves of this approximative verticality for rectifying afresh the pincers, and by proceeding thus for a series of successsive trials and corrections, the regulation will soon be completely effected. The small level perpendicular to the limb will then indicate if the verticality be preserved. We may even, by repeating these trials on different points of the limb, ascertain that the axis of rotation is exactly perpendicular to it; for if it were not, the positions of the plumb line would not agree on all the different radii of the circle.*

In order to appreciate the error arising from a small defect in the verticality, let the plane of the limb be produced to the celestial sphere, and the intersection will be a great circle, as H Z′ H, *fig.* 7. The most elevated point Z′ of this circle, will be the false zenith, indicated by the instrument, and the line Z′ O, drawn from the centre of the limb to this point, will be the apparent vertical, about which the zenith distances are measured on the limb. Now let O Z be the true vertical, in which case the angle Z′ O Z will be the inclination of the plane of the limb to it; and call this angle I. This being premised, if from the point O, the visual ray O S be drawn to any star, the true zenith distance will be the angle Z O S, which denote by Z. But the false distance, measured upon the limb, will be Z′ O S, which call Z′. The three sides Z Z′, Z S, Z′ S, will form a spherical triangle, right angled at Z′, and in which we shall have—

$$\cos. Z = \cos. Z' \cos. I;$$

from this Z may be found, Z′ being known. But the calculation must be made with minute accuracy, on account of the factor, cos. I. which differs very little from unity, since the inclination I. which may remain after the preceding verifications, is necessarily very small. Hence it is more simple to find the difference of the angles Z and Z′ approximately. In order to accomplish this, for the cos. I, let its value $1 - 2 \sin.^2 \frac{1}{2} I.$ be substituted, and we shall have

$$\cos. Z' - \cos. Z = 2 \cos. Z' \sin.^2 \tfrac{1}{2} I.$$

Rigorously, we might be satisfied with these precautions relative to the verticality, for, if we should find it deranged in an observation, the repelling screws of the instrument would be sufficient to re-establish it, according to the indication of the perpendicular level. But as this operation takes a little time, it is proper that it should be required as seldom as possible. Now, we should be forced to make it at each observation after turning the circle, if the column which supports the latter were not exactly vertical. For, let L M *fig.* 9 represent the limb, supposed vertical, in the first position of the instrument when it faced to the east, and A B the column inclined to the horizon : as the column

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Now, $\cos. Z' - \cos. Z = 2. \sin. \frac{1}{2}(Z + Z'). \sin. \frac{1}{2}(Z - Z')$. Substituting this value in the preceding equation, we shall find

$$\sin. \tfrac{1}{2}(Z - Z') = \frac{\cos. Z'. \sin.^2 \frac{1}{2} I.}{\sin. \frac{1}{2}(Z + Z')}$$

that is the sine of the required difference, which will always be positive when Z' is less than a right angle. Consequently, within this limit, the true distance always exceeds the observed distance; this ought to take place, since Z is the hypothenuse of the triangle. As the inclination I can never surpass a few minutes, the factor $\sin.^2 \frac{1}{2} I$, which is contained in the numerator, will always be a very small number, and the denominator $\sin. \frac{1}{2}(Z + Z')$ will be comparatively very great, even for a zenith distance of 1°: So that, below this term the difference of the arcs Z and Z' will be extremely small, and then we may suppose $Z = Z'$ in the second member, without apprehending any error; or, in other words, we may neglect the square of $Z - Z'$. We shall therefore find definitively,

$$\sin. \tfrac{1}{2}(Z - Z'') = \frac{\sin.^2, \frac{1}{2} I}{\text{tangt. } Z'}.$$

In order to investigate the accuracy of this formula, the results that it gives must be compared with those obtained from the rigorous formula, $\cos. Z = \cos. Z', \cos. I$. Let it be supposed that $I = 10'$, and $Z' = 1°$, sexagesimal, It will be found that the two formulæ agree very nearly. The approximation diminishes nearer to the zenith: and at last it ceases to be sufficiently accurate; for example, when Z' is nothing, it gives $Z' - Z$ equal to infinity, instead of

is fixed while the limb is passing from the east to the west, this last will take the direction L′ M′, which has the same inclination as before with regard to the column, but which is no longer vertical; wherefore it would be requisite to rectify the limb at each observation, after it has passed from one side to the other.

These inconveniences are avoided by rendering the column vertical, and it is brought to this situation by means of three screws adapted to the horizontal circle upon which the column is raised, and which serves for the base of the whole instrument.

These three screws denoted by V, V′, V″ in the *fig.* 10, are placed at equal intervals, so that the radii

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which the accurate formula then gives cos. $Z = \cos. I$, or $Z = I$, that is, all the error of verticality is transferred to the zenith distance, which in fact ought to be expected.

When the results have been obtained for any one inclination, as in the present example where $I = 10'$, and it is required to deduce from them the results relative to any other inclination I′, it will be sufficient to multiply the first results by the ratio $\frac{I'^2}{I^2}$ for it is evident that the values of sin. $\frac{1}{2}(Z - Z')$ are to each other as the squares of the sines of inclination when the zenith distances are equal; and as the angles $Z - Z'$ are very small, the arcs have also the same ratio.

From this we shall conclude, first, that the defect of verticality must be attenuated as much as possible; secondly, that the observations should not be made very near the zenith, where the influence of this defect upon the distances is most sensible, on account of the denominator tang. Z′. This last inconvenience never exists with respect to the polar star, by means of which latitudes are generally determined. Its distance from the zenith is out of the limits where the errors of verticality are considerable, at least, in all the habitable countries where the observation may take place. Finally, should the observations be made even at a few degrees distant from the zenith, it will be proper to ascertain the inclination of the plane of the limb, as exactly as possible, and to take it into the account by means of the preceding formula, which will attenuate the error, and may even render its influence altogether insensible.

CV, CV′, CV″, drawn perpendicularly to the axis of the column, make with each other angles equal to one third of the circumference. The process consists in first rendering one of these radii, CV for example, horizontal, and then turning the plane V V′ V″ about CV considered as an axis, so as finally to render it perfectly horizontal in all directions. Then the axis C C′, which by the construction is perpendicular to the plane V V′ V″ is necessarily vertical.

To accomplish this we use the larger level NN′, which is adapted to the column of the circle, and which serves to preserve the zenith in the revolving movements of the instrument. The limb is to be directed into the vertical plane of the screw V, and the level placed horizontally; then an azimuthal motion is given to the circle about the column CC′, and it is caused to make half a revolution, which brings the limb again into the vertical plane of the screw V. But then the two ends of the level have changed their positions with respect to that of the screw, the end which was south has become north, and that which was north has become south; and, since the rotation was made about the column C C′, the level has described a conical surface about the line C C′ as an axis. If this axis be vertical, that surface becomes a horizontal plane, and the level is not deranged; but, if the column be inclined to the horizon, the level can no longer be in a horizontal situation, and the inclination that produces its deviation from it, is doubled by the revolving motion. The half of this deviation is therefore corrected by turning the screw V in a proper direction, that is, by raising or lowering the point V, according to the indication of the level. Then the other half of the correction is to be effected by moving the level itself by means of its repelling screws, until it is brought to the point of equality, in which it

nearly. The azimuth being correctly read, turn the column half a circumference, so that the index, which at first answered to the azimuth A, now answers to the azimuth $A + 200°$;* and after it has been brought exactly into this position, fix it by means of its pressing screw, so as to render it invariable. By this operation, the extremities of the telescope have changed places; loosen it, and by sliding it upon the limb, bring it again to the point of direction precisely as if you wished to take the zenith distance of that point in an even observation. If the optic axis of the telescope be parallel to the plane of the limb, the point of direction ought to be found at the intersection of the wires. But if this axis have the least inclination, as it describes, by the revolution, a conical surface about the central axis perpendicular to the limb, the point of direction will no longer be found at the aforesaid intersection. It will depart from it either to the right or to the left. If the optic axis really approach the limb on the side of the object glass, the point of direction will appear to be removed from it. On the contrary, should that extremity of the optic axis recede from the limb, the point of direction will appear to have approached it, because the telescopes reverse the objects; and as the plane of the limb is always in the same vertical plane, the apparent deviation of the point of direction is double the error produced by the inclination of the optic axis. This is what is shown in *fig.* 11, in which A C B represents the vertical plane of the limb, L′ C O′ the first direction of the telescope towards the point O′, which answers to the centre of the wires, L″ C O″, the direction of the telescope after the revolution, and O′ O″ the deviation of the object O′ by the

* The instrument being supposed to be divided into 400°.

effect of the inclination of the optic axis upon the plane of the limb, double the deviation O′ B which represents the real effect of that inclination. *

When this error is to be measured, loosen the index of the azimuthal circle, so that the column may be turned, and bring the intersection of the wires upon the point of direction; the angle passed over by the index upon the azimuthal circle will be equal to the angle O″ C O′, and consequently double the error O″ C B, supposing the object to be in the horizon. If it be not, the angle passed over upon the azimuthal circle will be the horizontal projection of the real angle. The object has been

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* For ascertaining the error that would result from a defect of parallelism of the optic axis, conceive a visual ray to be drawn to the observed object, and to pass through the optic axis of the telescope. The arc between this ray and the vertical line will be the true zenith distance, or Z; but the apparent distance, or Z′, as read upon the limb, will only be the projection of the preceding upon the plane of the limb, by means of an arc of a circle, which measures the inclination of the optic axis; let I be this inclination; then the arcs Z, Z′, and I, will be the three sides of a right-angled spherical triangle, of which Z is the hypothenuse, and, consequently, in which we have

$$\cos. Z = \cos. Z' \cos. I.$$

This equation is the same as has been found for the verticality of the limb; therefore the same approximation may be obtained from it, which will always be sufficient, viz.

$$\text{Sin}\, \tfrac{1}{2}(Z - Z') = \frac{\text{Sin}^2\, \frac{1}{2} I}{\text{tang.}\, Z'}$$

This error for the optic axis is generally less than that for the verticality, as it is very easy to avoid an error of 1′ in the former, and the effect would be insensible in such zenith distances as it is customary to observe with the repeating circle. It also appears, that this effect tends to diminish the zenith distances, since the distance Z′, read upon the limb, is less than the real distance Z.

Should the optic axis have been badly regulated, or the objects observed at too great a distance from it, the observations may always be corrected by means of the preceding formula.

chosen in the horizon, that the operation might give that measure at once.

But, instead of measuring the error, if it be wished to correct it, move the micrometer towards the point of direction, until half the deviation be annihilated. If this bisection be made exactly, the optic axis will become parallel to the plane of the limb. But, as we are never certain of accomplishing this the first time, the verification is to be recommenced with the optic axis so regulated; and by a few trials, the parallelism will be obtained, with all necessary accuracy.—*(See the preceding Note.)*

I take this opportunity of remarking again, that the accuracy of astronomical processes is always founded upon a series of trials and successive approximations.—This is equally true for the results of astronomical calculations; and the same principle, transferred to other sciences, offers equally the surest means, I might say almost the only means, of arriving at a great precision.

By means of the verifications which have been described, the circle is completely regulated, and may immediately be used in observations. All these verifications are much longer to explain than perform by those accustomed to the use of the circle.

The repeating circle is not only useful in astronomical observations, it also serves in geodesic operations and in taking plans, for measuring the angles of position comprised between the objects. In order to render it more proper for this purpose, a second telescope is substituted for the larger level, moveable about the centre of the circle like the first, but placed on the other side of the limb; the limb is to be loosened and brought into the plane of the two objects of which the angle is required to be measured; for which purpose, there are repelling screws in all circles. In *fig.* 12, let S′ S″ be these two objects, and

C the centre of the circle. After bringing the upper telescope L′ to the point zero of the division of the circle, fix it there, and, without deranging it, turn the limb until the point of the object S′, upon which the visual ray is to be directed, is found at the centre of the wires. Then direct the lower telescope L″ upon the object to the left, which is here S″, and the optic axes of the two telescopes will comprise between them the angle S′ C S″, which is that of the two objects. But as the point A′, which answers to the second telescope, is not marked upon the limb, that angle cannot be read. This is supplied by doubling it, as is done to avoid reading off the plumb line in vertical observations. Without touching the telescopes, turn the limb, so that the lower telescope, which was directed upon the object to the left, be now directed to that on the right. Then the upper telescope will take the direction C L′, *fig.* 13, loosen it, the other remaining fixed, and bring it upon the object to the left, which will give it the position C A″ S″, *fig.* 14. In this movement it has evidently described the angle A C A″, exactly double the angle comprised between the two objects. Thus, half the arc marked by the vernier of the upper telescope, will give the required angle.*

* It is here supposed that the limb is divided from right to left, as in *fig.* 2. Then, by operating as directed, the upper telescope C L′ moves according to the order of the divisions. But if the circle be divided from left to right, the arc thus passed over is what the arc indicated by the vernier wants to be equal to the whole number of degrees marked upon the limb. In this case, it would be a little more convenient to direct the upper telescope to the object on the left, and the lower telescope upon that on the right, which afterwards requires the circle to be turned from L′ towards L″, *fig.* 12. in the contrary direction to that which we have supposed. At all events, in whatever manner we operate, there is not any difficulty in valuing the arcs.

If you now wish to have the quadruple angle, leave the telescopes fixed; and, after turning the limb until the upper telescope is brought upon the object to the right S′, loosen the lower telescope, and bring it upon the object to the left, S″, *fig.* 15; then the circumstances are exactly the same as in the first observation; only the point of departure of the upper telescope upon the limb is not at A but at A″; that is to say, at the extremity of the double arc passed over in the first pair of observations. Thus, by operating again in the same manner as before, a new double arc will be added to the former, and if the observations be thus continued, any multiple of the angle will be obtained that may be wished. When it is thought, from the number of observations, that a sufficient accuracy has been obtained, the verniers are to be read, and, by dividing the total arc by the number of observations, we shall have the single arc. This proceeding evidently possesses all the advantages arising from the repetition that have been observed with regard to vertical angles.*

See now the precautions and verifications which it requires.

The first is to render the optic axes of the telescopes

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* We find in the Memorial Topographique et Militaire, the following specimen of calculation for angles between terrestrial objects.

Example.

That is called the first index which is on the right of the eye-glass of the telescope, and which usually carries the repelling screw. The second, in following the order of the divisions, will be that which is towards the object glass of the upper telescope. The third is opposite the first, and the fourth to the second.

Suppose that the first index being fixed *at zero,* the three others mark the following numbers, the instrument being divided into 400 degrees.

parallel to the plane of the limb: nothing is more easy. The process, with regard to the upper telescope, has been explained: when this is regulated, we direct it to a distant object, and bring the axis of the other telescope to the same point. The optic axes of the telescopes are then parallel to each other; they are therefore both parallel to the plane of the limb, since one of them was before brought into that situation.

Though the repeating circle has all the advantages which we have explained, it ought not to be dissembled, that some defects of execution on the part of the artist might destroy all its accuracy; and this is so much the more important to consider, as the defects in question are very easily discovered; as it is very easy, when we foresee them, to elude their influence; and fi-

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1st	2nd	3rd	4th
0.° 000	100.° 046	199.° 988	299.° 969

the sum of which, by rejecting the units, is =0.° 003; and that we have observed.

	Multiple angles.		*A or single angles.*	
2 A	90.°	842	45.°	421
4 A	181.	684	45.	421
6 A	272.	527	45.	421166
8 A	363.	379	45.	422375
10 A	454.	225	45.	4225
12 A	545.	073	45.	42275
14 A	635.	921	45.	42298
16 A	726.	772	45.	42325
18 A	817.	618	45.	42322
20 A	908.	47	45.	4235
22 A	999.	32	45.	423636
24 A	1090.	164		
	.	212	$\frac{.666-.003}{4}$	
	.	154		
	.	136		
24 A	1090.	16575	45.°	423579

nally, as, without proper precautions, we should be exposed to great errors, of which nothing may warn us.

From what has been said at the beginning of this chapter, upon the construction of the repeating circle, it is evident that the only defects in it which we have to fear are those which may oppose the principle of the repetition, by producing a constant error at each observation. In fact, a similar error, accumulating with the number of observations, and not being susceptible of

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Having taken the sum of the numbers marked by the four indexes, and rejected the units, the angles given by the successive observations are to be written in a column, reading only one index at each observation; from these multiple angles the single angle shall be determined, which will also serve to show the terms of the series, and to make known if any error has been committed in the observation. When ten or twelve double angles have been taken, the error from the division will be sufficiently attenuated, and the four indexes are then to be read; the numbers they give, omitting the units, are to be written one under another, and their sum taken, which in the example before us is 0.666, from which 0.003 is to be subtracted, that is to say, the sum of the errors given by the indexes when the first is at the *nought* of the division. The fourth of this difference is the decimal part of the number of degrees given by the last observation. Thus, in the above example, $0.666 - 0.003 = 0.663$, the fourth of which $= 0.16575$, therefore twelve times the double angle $= 1090^\circ.16575$; dividing by 24, gives the single angle $= 45.^\circ 423579$.

We might, strictly speaking, dispense with reading all the observed angles, and read only the last; but, if we have time, it will be preferable to read them all, in order to know the terms of the series, as has been said, and be able to retake them in case of an error.

It is always proper to read the numbers marked by the four indexes prior to commencing the operation for measuring an angle, as they may mark different numbers at different times; and this may happen from a slight eccentricity of the indexes, occasioned by conveying the instrument from one place to another, or even by the different expansion of the brass rulers which connect the indexes.

compensation, since we always suppose the instrument of an invariable form, would necessarily be found entire in the mean zenith distance, which, in this respect, would not be more exact than that derived from a single observation.

Of this kind are the errors due to the inclination of the optical axis and to a defect in the verticality of the plane of the limb. We have given the methods of measuring and destroying those errors, or of attenuating them sufficiently to render their influence absolutely insensible. But there may be still other causes of constant errors in circles, to which we have not yet attended. The first is the error of *centrage*. It takes place, if the upper telescope, in turning about the centre of the limb, can play about the axis which supports it. In order to investigate the errors in the zenith distances which would result from this cause, let us commence with the first position of the telescope at the beginning of the observations, which is represented by O C′ A, *fig.* 16; then the ring M N, pierced at the centre of the index which carries the telescope, has descended by its weight upon the axis C′ M, about which the telescope turns; and the ring touches the axis in M, at its most elevated point. Things remain the same in the second position of the telescope after turning the circle, in order to pass to the even observation, *fig.* 17 and 18. Only the point of contact of the ring has passed from M to N, upon the opposite part of its circumference. Consequently, the arc A Z A′ passed over by the index upon the limb, *fig.* 18, is greater than the true angle a C′ a′, and the difference is equal to Aa + A′a′ or 2 Aa. But a contrary error, and more considerable, takes place in passing from the second observation to the third. In fact, at the end of the se-

cond, the telescope has the position O C′ A′, *fig.* 18. In the revolving movement of the circle, the telescope takes the direction O C′ A′, *fig.* 19. It preserves the same inclination to the horizon; the point of contact N remains the same, and the point A′ likewise. But, when the limb is afterwards turned vertically about its centre C′, in order to bring the telescope again towards the heavenly body, and to give it the direction O C′ A″, as in *fig.* 20, the point of contact of the ring, upon the central axis, is no longer at the same point N. The telescope falls by its weight, and the contact returns to the side of M, in the opposite part of the circumference. This movement causes the point A′ to approach A, because the eye-glass, denoted by O in the figure, is fixed by means of its pressing screw. The telescope therefore turns about the point O as a centre. The object glass, in descending, returns upon the limb by a quantity A′ A″, quadruple of A′a′; and, as the double arc concluded from the first pair of observations was too great by 2 A a, the quadruple arc concluded from the two consecutive pairs, is too small by the same quantity. It is clear that the same effect will always be produced upon each pair, and that for equal zenith distances, it will always be in the same direction by the same quantity, since the ring of the index and the central axis are both circular. This is therefore a cause which will always tend to diminish the measure of the zenith distances. It is so much the more to be feared, as the radius of the ring and that of the axis are generally very small, so that even the least play between them considerably increases when transferred to the limb. It is therefore of the utmost importance that the juxtaposition of these two pieces should be observed with extreme care, and for this purpose, the axis of steel, which carries the index, should be sufficiently produced, that

the artist may adapt it exactly to the ring. Unfortunately, it appears that sufficient attention has not yet been paid to this, except in the last circles that have been constructed.

Another effect, absolutely similar to this, is produced, if the pressing screw, designed to fix the index upon the limb, has not sufficient power to retain it invariable; for, in passing from the second observation to the third, or, in general, from an even to an odd one, the telescope might slide by its weight, in consequence of the motion which is given to the limb; and then the point A′ is also displaced in one direction or the other, so as either to augment or diminish the zenith distances. This displacement will only have an influence in the passage from odd to even observations, in which the fixity of the telescope upon the limb is indispensably requisite.

Finally, an effect of the same kind is also produced if the repelling screw which moves the telescope upon the limb has the least play in its nut or its collar. For, without entering here into too minute details on the construction, this screw may be considered as intended also to retain the telescope fixed in passing from even to odd observations. If the screw be too loose in its nut, the telescope which it moves might absolutely take the same movements as the screw; now, as the telescope is necessarily reversed in passing from even to odd observations, the repelling screw, which is connected with it, takes also opposite positions, as is shown in *figs*. 18 and 20, in which this screw is denoted by V. Consequently, if it have any play in the nut which retains it, there will be one of these two positions in which the telescope will no longer be properly supported. Thus it will turn in the direction in which it is drawn by gravity, that is, the point A′ will descend if the end which carries the

object glass be heavier than that where the eye-glass is fixed; should it be the contrary, it will rise: in the first case, the zenith distance given by the first pair of observations, will be diminished; in the second case, it will be increased.

In vain might we wish to guard against these errors by reading with care the verniers in the two opposite positions of the telescope, as they could not be thus perceived unless they are enormous. Fortunately, they may all be corrected by a very simple means; this is, by observing the zenith distances of stars to the north and south of the zenith; for, if the circle give the distances too great or too little, the sum of these distances will also be too great or too little by a double quantity. Now this sum is the difference of the polar distances of the observed stars; thus, by taking this difference in very exact catalogues of the stars, it will be seen if the circle has any error, and in what direction the error lies. Unfortunately, we only know perfectly at present the polar distances of those stars of which the two passages can be observed. There is much uncertainty respecting the others, and it would be a work of the greatest utility to determine the declination of the most brilliant of them exactly, by means of their meridian altitudes observed with the repeating circle, in a place such as Paris, the latitude of which has been exactly determined by a multitude of passages of circumpolar stars. However, without waiting for the execution of such a work, we may still supply the knowledge of the polar distances, when we do not want an absolute latitude, but only the difference of latitude of the two extremities of an observed arc, as generally happens in geodesic operations, which are those in which the greatest accuracy is required. It is then sufficient to repeat at these two extremities the observations of the zenith distances of the same stars to the north and south

of the zenith. The sum of these opposite distances is to be taken for each station, in order to have the measures of the same celestial arcs. Then, if any difference be found, its half must be added to the distances observed in the station where the sum of the distances is the least. By this means the error of the circle will be the same at the two stations, and the difference of latitude will be correct. But the absolute latitude might have an error of the whole quantity corresponding to the absolute error of the circle.*

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* Let N be the true zenith distance of a star situated to the north of the zenith, in the most northerly station. Let S be the true zenith distance of a star situated to the south, and observed at the same station. Call N' and S' the analogous quantities for the most southern station. These would be the quantities observed if the circle had not any error. Then the difference of latitude would be $N'-N$, or $S-S'$; and the celestial arc $A=N+S$; $A'=N'+S'$, would be the same in both stations, so that we should find $A=A'$. It will not be the same if the circle have had a constant error e in the first station, and the error e' in the second. For then the observed zenith distances would be $N+e$, $S+e$; $N'+e'$, $S'+e'$, and when these are combined in order to find the difference of latitude, we shall have, for the northern stars, $N'-N+e'-e$; for the southern stars, $S-S'+e-e'$. The errors affect these results in contrary ways, hence, they disappear by adding, and their half sum $\frac{N'-N+S-S'}{2}$ will give the true difference of latitude, notwithstanding the constant errors at each station.

But if it be wished to value the difference of these errors, that is, the quantity which the constant error of the circle has changed, by transporting this instrument from one station to the other, nothing is more easy; for, by taking, at each station, the sum of the zenith distances, in order to obtain the celestial arc, we shall have

$$\text{in the first,} \quad A=N+S+2e,$$
$$\text{in the second,} \quad A'=N'+S'+2e';$$

and as $N'-N$ is equal to $S-S'$ if these two equations be subtracted the one from the other, there will be found,

$$A-A'=2e-2e', \text{consequently } e-e'=\frac{A-A'}{2}$$

It appears evident that the causes of the errors above explained have produced the small differences which have sometimes been remarked between the absolute latitudes observed at the same point with different circles, or by different observers, although the particular results agreed with each other. The corresponding observations at the north and south would have corrected the effect of these differences, if that means had been thought of sooner; it appears indispensable to employ it henceforth whenever the greatest possible accuracy is required.

In order to complete this chapter, I conceive it will be useful to explain accurately the theory of air levels; as many readers may not have very precise ideas respecting these instruments, though they are very frequently used in astronomy and physics.

We generally call a tube of glass closed at both ends, and partly filled with a liquid, as water, alcohol, or ether, an air level. The space which is not filled with the liquid is occupied by air, or at least by the vapour of the liquid; and as this, by virtue of its gravity, always tends to occupy the lowest part of the tube, and to preserve a horizontal surface, it results from it that the bubble of air is thus displaced, and ascends to the top of the tube. Consequently, its movements indicate the variations in the inclination of the plane upon which the level is placed.

Let us first examine the case in which the bubble of

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that is the difference of the errors. Add this to the zenith distances, $N' + e'$, $S' + e'$, observed at the second station, they become $N' + e$, $S' + e$. They will therefore then be comparable to the observed distances in the first station, and, by combining them, we shall have $N' - N$, or $S - S'$, for the true difference of the latitudes.

air is so small, that it may be considered as a point. If the interior of the tube were perfectly cylindrical, there could be, strictly speaking, only one single position of the level in which the bubble would remain stationary in the middle of the tube; this would be the horizontal position; the least inclination would displace the bubble, and carry it wholly to one of the extremities. It is therefore necessary, in order to avoid this inconvenience, that the interior of the tube should be slightly arched; then the bubble will take such a position as will cause its middle to answer to the point where the tangent of the interior curve of the tube is horizontal. If the inclination change, the bubble is displaced, and is carried to that point of the tube which has become the most elevated; that is, where the tangent has become parallel to the horizon.

Among all the curvatures that may be given to the tube, the most advantageous is that of the circle, because the displacement of the bubble then immediately measures upon the tube the variations of inclination. In fact, let N S, *fig*. 21, be an indefinite arc of a circle which represents the longitudinal section of the level. Let C be the centre of the arc, and C A a radius directed according to the vertical line; in this position of the level, the bubble of air, which we still suppose to be extremely small, will be placed at A. But if the inclination changes, the radius C A will be inclined, as in *fig*. 22, and another radius, such as C A′, will become vertical; at the same time the bubble will be displaced and carried to the point A′. The arc A A′, which it describes upon the level, is the measure of the angle A C A′, or of the change of inclination.

In all this it is supposed that the level is only a circular physical line. This is not possible in practice;

but the advantages of the circular figure may still be preserved by considering the interior surface of the level as a ring generated by the circle D D', *fig.* 22, the centre of which moves upon the circular direction N S. Such a surface evidently preserves all the properties of the system which we have considered. As to the manner of giving this curvature to tubes, it consists in wearing out their interior surfaces by friction, with metallic rods which are successively introduced at both ends, until it is found, after numerous trials, that the motion of the air bubble agrees exactly with the changes of inclination. In order to see when this condition is fulfilled, the level is placed on an inclined plane of a determinate length, the inclination of which may be rendered variable by means of a vertical screw, the distance of whose threads is known. The head of the screw supports a dial plate divided into equal parts, and an index like the screw of a micrometer. In turning the screw a known quantity, it is easy, according to the preceding data, to calculate the resulting variations in the inclination of the plane, and it is then seen if the motion of the bubble in the tube is exactly conformable to them. To do this we commence by drawing upon the glass tube equal divisions through its whole length.*

For example, by submitting to this proof a level constructed by Fortin, for observations of the latitude at Dunkirk, it was found that the air bubble passed over three millimetres (about .118 of an English inch) upon the tube, for a change of one sexagesimal second in the

* The practical explanation of these proceedings is found in a very good memoir of M. de Chezy, upon the construction of levels, inserted in the memoirs of the Academy of Sciences and learned Foreigners.

inclination, and it preserved this motion with perfect regularity through the whole extent of the divisions. This result may also be verified in a more exact manner when the level is adapted to a repeating circle, by observing a distant object. To accomplish this, place the limb in a vertical position, and direct one of the feet of the circle in the azimuth of the object; bring the upper telescope to zero, and turn the limb until the object comes to the centre of the wires; then place the great level so that the bubble may rest very near one of its extremities. This done, without touching the telescope, turn the level and the limb together in their vertical plane, by means of the repelling screws, until the bubble is brought to the other extremity of the tube, so that it thus passes over a great number of divisions; then the object no longer answers to the centre of the wires, and in order to replace it, the telescope must be moved on the limb. The arc which the telescope thus describes being read, the change of inclination corresponding with the number of divisions that the bubble has passed over will be known; and, consequently, by dividing this inclination by that number, we shall have the value of each of them.

The arc passed over in an observation of this kind being always very small, a single observation will not give it with sufficient accuracy, because of the errors that may be committed in reading the verniers; but the principle of repetition may be applied here. In fact, the first observation being made, and the telescope brought again upon the object, displace the great level, and bring the bubble to the other extremity of the tube; then, without touching the telescope, turn the limb vertically, until the bubble return to the opposite extremity; and as this movement will displace the object,

bring it again to the centre of the wires, by moving the telescope; by this means, the telescope will describe a new arc, which, being added to the former that it had passed over, circumstances will be precisely the same as at the time of the first observation; wherefore the same operation may be repeated, and even a third or fourth observation made, so that the total arc passed over may be large enough to allow the errors of the extreme readings to be a very small aliquot part of it. This arc, corresponding to 1000 or 1200 parts of the level, will give, by the division, the value of one of them with extreme precision. This proof is so much the more necessary, as it indicates the value of the divisions on the level when it is in its place, which value may not be the same as when the level was free and simply placed upon an inclined plane. For the mountings, in which the level is generally set, may bend it and change its curvature, especially if it has a considerable radius, as in that used at Dunkirk, the radius of which was 619 metres, or about 677 English yards.*

In all the preceding explanations, it has been supposed that the air bubble was sufficiently small to be regarded as a point. This consideration was useful for simplifying the reasoning, but it is properly avoided in practice; for experience shows that a bubble so small moves with great slowness, and that the least obstacle, the least irregularity in the tube is sufficient to

* This appears from the motion of the bubble in the level. Upon this an arc of one sexagesimal second is expressed by three millimetres; now, it is known from trigonometrical calculations, that the radius expressed in terms of the arc is equal to 57° 17′ 44″. 8 of the sexagesimal division, or which is the same thing, to 206264″ 8. Since each second amounts to three millimetres, the radius will be three millimetres multiplied by 206264. 8, or 618,7944 metres, or nearly 677 English yards.

stop it; on the contrary, the bubble is made very long, because it has been observed, that the longer it is, the more sensibly; that is, the more quickly, it comes to an equilibrium.

This phenomenon depends upon the reciprocal attractions of the liquid and the glass: it is of the kind of those which we call capillary phenomena, because they were at first observed in very small tubes, in which it is known that liquids which moisten the glass, rise above their natural level, and form a small column terminated by a concave surface. The edges of this surface, composed of molecules adhering to the interior of the tube, rise along it, and their inclination depends on the greater or less action of the tube upon the fluid, as well as on the fluidity being more or less perfect. An analogous effect is produced in levels, upon the extremities of the bubbles which they contain. These extremities are raised along the tube, by which means the bubble is rounded in the places where the surface of the liquid touches the glass; and more so at both ends than along the sides. The bubble is also more rounded in a narrow than in a wide tube. This agrees with the nature of capillary forces, which, commencing at the surface and having a sensible action only at a very small distance, have an effect so much the more intense, as the surfaces which exercise them are nearer to each other. For the same reason the elevation of liquids, in narrow and vertical tubes, increases as their interior diameters diminish, so that the liquid column thus raised is reciprocally as the diameter, as both experiment and calculation agree in proving. From this it may be conceived that the effect of these forces ought especially to be considerable upon a very small bubble, about which the fluid forms a concave surface of a very small radius. Then the least asperity of the tube may considerably change

the attractions that determine this form, and lengthen the bubble in one direction more than in another, or even absolutely stop it; instead of which, these effects will be much less upon a long bubble and in a large tube, in which the raising of the fluid, by the effect of capillary action, will be much less. Such are the advantages of great levels, similar to those which Fortin has constructed for our circles. But, in order that this length may be useful, the interior of the tubes must be well worked, and exactly bored according to the circular curve; precautions which artists do not commonly take, contenting themselves with the natural curves which tubes of glass always take when they are made.

The bubble of the level having a sensible dimension, its middle point is regarded as the most elevated point of the tube, and it is the motion of this middle point that determines the changes of inclination. In order to express and measure these changes, the tube is divided in all its length: the zero of the division answers to its middle, and, for determining the position of the centre of the bubble, half the sum of the divisions which limit its extremities is taken. This is strictly true if the tube be well bored, according to the annular form, for then, its form being symmetrical in all its parts, that of the bubble will necessarily be the same. We might even, if we wished, take one of the extremities of the bubble as the index of its motion, but the other method is more advantageous, because it attenuates the small irregularities of the tube by dividing them between the two extemities of the bubble. At all events, it is certain that both these proceedings are equally incorrect if the tube be badly worked within.

It is the common custom to place the zero of the division, as we have said, at the middle or very near the middle of the tube: which is, in fact, more commo-

dious, as the numbers are then more simple. But this is by no means necessary; we might begin the divisions at one of the ends, and half the sum of the divisions, marked by the extremities of the bubble, would always give the position of its middle point: in general, at whatever part the centre of the graduation was placed upon the tube, the level would serve equally.

The extreme perfection that has been given to levels of late, has permitted to fix them to repeating circles; and to determine, by their variations, the changes of inclination in the axis. By this means we avoid the necessity of replacing the level for correcting these changes, as was done before, which required time, and increased the difficulty of the observations by the union which was necessary between the observer who directed the telescope, and he who rectified the level.

In this new disposition, the level is firmly attached to the vertical column about which the circle turns. It is placed perpendicularly to the direction of that column, and parallel to the plane of the limb. The limb itself may also be firmly attached to the column, or rather to the vertical plane which the latter carries, and upon which the limb is fixed by means of two strong screws. In order to make the first observation, we begin as usual by bringing the telescope to the zero of the division upon the limb, or, which amounts to the same thing, by reading correctly the points of the division marked by the verniers: these points serve for the origin of the arcs which the telescope describes. Next, after loosening the limb, and turning it until the telescope is directed towards the heavenly body, we fix it in that direction, and use the repelling screws to bring the body exactly to the wires. At this instant one of the observers reads the arc, and the other the level. The circle is then caused to make

half a revolution in the usual manner about the vertical column in order to pass to the second observation. This time the limb must remain fixed, but the telescope is to be loosened and brought again to the heavenly body. If the column, or rather if the axis of rotation of the circle, still remains exactly vertical, as we have supposed it was in the first observation, the level will not have been deranged by the revolving movement of the circle; and the arc passed over upon the limb will be double the zenith distance without any correction. But if the axis were not perfectly vertical, or if it have taken any inclination in passing from the first observation to the second, as the circle has revolved about it, the bubble of the level will not correspond to the same points of the tube after the revolution. This is the reason why the observer again reads off the level in this second position, as he had read it in the first, and by means of these two readings the inclination of the axis of rotation on the vertical is found.

To conceive how this is done, let the respective positions of the level, and of the axis of rotation of the circle in the two observations, be carefully examined. Let P A, *fig.* 23, be the direction of that axis; and A Z S an arc of a circle representing the longitudinal section of the level. Let C be the centre of the level, which will not be placed in the prolongation of the axis of rotation, unless that axis be perpendicular to the level. The angle C A P will be the inclination of the axis of the level to the axis of the circle, an inclination which we call I'. If through the foot of the circle, or through the point P, the vertical line P V be erected, the angle V P A, which is denoted by I, will be the inclination of the axis of rotation of the circle to this vertical line. Finally, if

from the point C, the centre of the level, the vertical line C Z be drawn, meeting its circumference in Z, the point Z will be the centre of the bubble, since this centre ought always to be found at the most elevated point of the level; and the arc A Z, read upon the circular division of the level, will be the measure of the angle Z C A, that is, of the inclination of the radius C A to the vertical line. Now, by prolonging the direction A P of the axis until it meet the vertical line C Z in D, it is evident that the exterior angle Z C A of the triangle C A D is equal to the sum of the two interior angles C A D and C D A; the first has been denoted by I′; as to the second, it is equal to V P A, or to I. Thus, by substituting for these angles their values in parts of the level, we shall have $AZ = I + I'$.

Now, if the circle be returned as in *fig.* 24, as the rotation will be performed about the axis A P, the radius C A of the level will describe a conical surface about this axis; so that, after the revolution of the circle, the inclination C A P will be the same, but in an opposite direction; that is, it will still be equal to I′. Besides, if through the new position of the centre C, the vertical line C Z′ be raised, the point Z′ will be the centre of the bubble in the new position of the level; the angle Z′ D′ A will still be equal to I, and by a reasoning similar to the above, we shall have $AZ' = I - I'$.

Thus, the arc A Z, in the first reading, is the sum of the inclinations of the axis of the circle to the vertical line, and the radius of the level; while in the second reading, the arc A Z′ is equal to the difference of the same angles; from which it follows that the inclination of the level to the axis of the circle is equal to the half difference of these arcs, and the inclination of the axis of the

circle to the vertical line is equal to their half sum; that is to say, we have

$$I' = \frac{AZ - AZ'}{2}, \qquad I = \frac{AZ + AZ'}{2}.$$

It still remains to find the value of these arcs A Z and A Z'. This would be easy if the origin of the divisions of the level was exactly at the point A, in the prolongation of the axis of rotation; for then the simple reading would express their values. But, whatever may be the origin of the divisions, it is easy to obtain these values from the two readings combined, and they almost immediately give the correction due to the inclination of the axis.*

In order to perceive the effect of this inclination upon the zenith distances, let C, *fig.* 25, be the centre of the

* Let M, *fig.* 23, be the origin of the divisions of the level, or the position of the point zero on the tube; call A the distance A M, from that origin to the intersection of the level by the prolongation of the axis of rotation; let 2 L be the whole length of the bubble, and denote by N and S the co-ordinates of its North and South extremities, that is, the distances of these extremities from the origin M of the division, or the arcs M N, M S. This being done, the point Z being supposed to be the middle of the bubble, we have, in this first position of the level and of the limb,

$$S = AZ + L - A, \qquad N = AZ - L - A.$$

Let the limb be returned in order to pass to the second observation, *fig.* 24. In this movement the point M will pass to M' on the other side of the axis. Let Z' be the centre of the bubble in this new position, and call also N' and S' the co-ordinates of its North and South extremities, that is, the arcs M' N' and M' S', we shall then have

$$S' = AZ' + L + A, \qquad N' = AZ' - L + A;$$

from these expressions we obtain

$$AZ + AZ' = N + S' \qquad AZ + AZ' = N' + S$$
$$AZ - AZ' = N - N' + 2A \qquad AZ - AZ' = S - S' + 2A$$

but $\frac{AZ + AZ'}{2}$ is the inclination of the axis of the circle to the vertical towards the North, which has been denoted by I; and $\frac{AZ - AZ'}{2}$, is the angle formed by the axis of the circle with the

limb, C Z the vertical line, C S the visual ray drawn to the heavenly body to be observed ; the angle S C Z will be the true zenith distance. Now let C P be the direction of the axis of rotation of the circle, which is sup-

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radius of the level; which angle has been called I'. We have, therefore, the following values for these quantities,

$$I = \frac{S' + N}{2}, \text{ or else } I = \frac{S + N'}{2};$$

$$I' = \frac{N - N'}{2} + A, \text{ or else } I' = \frac{S - S'}{2} + A.$$

In order to obtain a greater degree of symmetry in the calculations, it has been supposed that the point M, which is the origin of the divisions, fell without the bubble on the Northern side. The bubble, however, is generally of such a magnitude, that this origin falls between its extremes; and when the level is horizontal, the point M is nearly in the middle of the bubble. Then the distance N, from the origin of the divisions to the Northern extremity, must be considered as negative in the above formulæ. The same also for N'; and with this modification the preceding expressions will become

$$I = \frac{S' - N}{2}, \qquad I' = \frac{S - N'}{2};$$

$$I' = \frac{N' - N}{2} - A, \qquad I' = \frac{S - S'}{2} + A.$$

We shall limit ourselves to examine the first two, which give the inclination of the axis; the two others, containing the distance A, which it is impossible to know, cannot be applied. But fortunately the angle I', which they determine, is not necessary to the observations. Now, it is evident that N and S' are the co-ordinates of the same physical extremity of the bubble, before and after the turning of the limb; for, if this extremity were North in the first observation, it is South in the second. It is the same for N' and S; these are the co-ordinates of the other end of the bubble. From that results the following rule:—Observe, upon the division of the tube, the two numbers which correspond to the same physical extremity of the bubble in the two consecutive observations. Half the difference of these numbers will express the inclination of the axis towards the end which has been considered as negative, that is, towards the North, if this end was directed to the North of the zenith in the first observation, and towards the South, if it was directed to the South.

This rule may be expressed in a manner still more simple, by referring it to the positions which the level successively takes, relative

posed to be inclined to the vertical on the side of the heavenly body; this axis, prolonged to Z', will determine the apparent zenith about which the limb turns. Now, it is to this apparent zenith that we refer upon the

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to the observer who reads its divisions. Suppose, that in the first observation the *South* end of the bubble is on the left hand of the observer; in this case, the *North* end will be on his right. Now, the limb is turned in order to pass to the second observation, and he who reads the level also turns in passing to the other side of the limb. Then the left extremity of the bubble is still the same physical end as before; only, instead of being directed to the South of the zenith, it is directed to the North. Thus, if we denote by L; R; L'; R'; the left and right extremities of the bubble in the two observations, we shall have $S = L$; $N = R$; $S' = R'$; $N' = L'$. By substituting these letters in the formula which gives the inclination of the axis towards the end N or R, we shall have,

$$I = \frac{R' - R}{2}, \text{ or else } I = \frac{L - L'}{2},$$

under this form the interpretation of the formula does not offer any uncertainty; and the application of it may be rendered very simple by the following disposition. Write the readings of the two extremities of the bubble successively in two columns, which call L and R, that is to say, left and right, as below.

Left		*Differences.*	*Right*		*Differences.*
142 130	+	12	136 148	+	12
142 130	+	12	136 148	+	12
144 134	+	10	134 144	+	10

The first contains the successive values of L, L', for each pair of observations; and the second, the values of R, R'. Thus, by taking the differences of these values for each pair, we shall have $L - L'$, and $R' - R$. And if the half of these values be taken, it will give the inclination I of the top of the axis towards the side R of the bubble, or towards the side R of the zenith.

Here, for example, each of the two first pairs, gives $\frac{R' - R}{2} = +6$; that is, the top of the axis is inclined 6 parts towards the side R of the zenith; consequently, towards the North if the end R is turned

limb all the observed distances; for the point of the limb that remains fixed in its revolution is the point A', situated in the direction of the axis of rotation of the heavenly body, instead of which, it should be the point A, situated in the prolongation of the vertical line. The

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to the North of the zenith. If the observed heavenly body is situated on the same side, as in *fig.* 24, the zenith distances will be too little by 6 parts of the level; wherefore they must be increased by that quantity. This would be two sexagesimal seconds in the level used at Dunkirk. But if the observed heavenly body were situated on the South of the zenith on the side S, and that the level had been read as is here supposed, the inclination towards N would augment the zenith distances, and it would be necessary to subtract it. The same result would be found from the *left* end of the bubble; for it gives $L = 142$, $L' = 130$, consequently $I = \frac{L-L'}{2} = +6$.

The third pair of observations presents a little change in the inclination, for we only find $I = +5$. We thus form all the differences relative to each pair of observations, and take the mean, which is the correction to be applied to the mean distance deduced from the extreme readings of the observed arc. In this addition, each of the differences $R'-R$ or $L-L'$ must be written with its sign, for some of them may be positive and others negative, if the level experience any derangement in the course of the series; and this will almost always happen when the inclination I is very small. If the tube be perfectly bored, and if the temperature do not vary, the length of the bubble remains constant, and it will be sufficient to observe one of its extremities: but as we can never be certain of absolutely avoiding all these variations, it is always proper to observe the two extremities of the bubble, and to take the mean difference.

There is an undoubted advantage in using constantly the same method of performing practical operations which are to be frequently repeated. Wherefore it is proper always to read off the level in the same manner; for example, in placing ourselves so that the side R may be directed towards the heavenly body which is observed; then, if $\frac{R'-R}{2}$ or Z is positive, it is to be added to the zenith distance; if it be negative, it must be subtracted. All is then reduced to this very simple rule, and there will never be any mistake committed respecting the direction of the inclination.

error of each distance is therefore equal to Z C Z′; that is, to the inclination of the axis to the vertical. It diminishes the zenith distances, when the top of the axis inclines towards the heavenly body, as in *fig*. 25; on the contrary, it increases these distances when the axis inclines from that body, relatively to the vertical line, as in *fig*. 26. Thus, in the first case, the correction determined by the level, ought to be added to the zenith distance; and in the second, it ought to be subtracted from it. The verticality of the axis was not necessary in the old circles, because the limb was rectified after the revolving movement, by means of the repelling screws, so as to bring the level to its primitive position. This is not possible in the new circles, since the level is not fixed immediately to the limb, but to the column of the instrument. It is true, we might at once raise or depress, according to circumstances, the screws adapted to the base of the column, and which serve to render it vertical. But it is a thousand times more simple, more commodious, and exact, to observe the deviation of the level, and to correct the effect in the observations; and so much the more so, as neither the calculation nor the application of this correction offers any difficulty.

This method only supposes, as a rigorous and indispensable condition, that the limb of the circle remains firmly fixed to the axis of rotation in the two consecutive observations, without the pincers, which retain it, permitting the least movement. If this condition were not fulfilled, the point of the limb which answers to the prolongation of the axis of rotation, would not remain the same in the two consecutive observations, and this change would produce an error impossible to appreciate. The most minute care must, therefore, be taken in making the instrument, that the point where the axis of ro-

tation and the limb are connected together be rigorously fixed; this is what our excellent Fortin has obtained by fastening the limb, by means of a strong pincer and two pressing screws, against a vertical plane of brass, which is itself fixed to the axis and supports the limb, as has been said.

It may happen, and indeed it commonly does happen, that the axis of the circle does not strictly preserve the same inclination throughout the whole duration of a long series of observations; but the level indicates all these variations, and as it is read off at each observation, it determines the true value of each correction: but only the mean of these corrections enters into the medium result.

When a long series of observations ought to be made, for example, when it is wished to determine a latitude with the greatest accuracy, care should be taken to calculate, every day, the value which the inclination of the axis has had; and as this inclination can be varied at pleasure by means of the repelling screws adapted to the base of the column, it should be so arranged, that its successive values, on each side of the zenith, may be nearly equal. Then the positive and negative corrections of the level compensating each other, we are independent of the values of its parts, or at least, if the compensations are not strictly exact, the number of parts of the level which remains in the mean result is so small, that the error that may be committed in their valuation is absolutely insensible. This is the method which we used in the observations at Formentera and at Dunkirk; as to the measure of the parts of the level, and the determination of their values, the operations have been explained at page 37.

Description and Use of BORDA'S *Reflecting Circle.*

THE following description of Borda's reflecting circle and of its use, is extracted from the second edition of Biot's *Astronomie Physique*, where it is given by DE ROSSEL, late a captain in the French navy.

We are indebted to the learned *Borda*, who by his works has contributed so much to the progress of navigation, for the best reflecting instrument. His first intention only was to improve the circle of *Tobias Mayer*, a professor at Gottingen; but the improvements he made imparted to it so many advantages, that he may be justly regarded as the inventor of the reflecting circle now in use.

The reflecting circle (*figs*. 28, 29, 30, and 31,) consists of an entire circle; but as it is calculated to give the angles that the rays reflected from a mirror make with each other, half degrees must be reckoned for degrees, and the circle is divided into 720 parts instead of 360. The large mirror L G is placed at the centre of the instrument, and turns about it by means of the index C B. The small mirror I F, instead of being in a fixed position, as in the sextant, is placed at the extremity of a second index O P, moveable about the centre of the instrument, and carrying the telescope O at its other extremity. The position of the telescope O therefore remains constant with respect to the small mirror I F; but when the index O P is moved, this mirror changes its position relative to the large mirror, and takes different degrees of inclination with it. Suppose the index B C, which is called the index of the large mirror, to be at the point zero of the graduation, and that the index P O of the small mirror is fixed by a pressing screw in such a position that the two mirrors are parallel; then, if the

index of the large mirror be moved from B to D, and an object seen directly in the telescope O be made to coincide with the image of another object reflected by the large mirror, it is evident that double the arc B D, or the number of degrees and minutes marked by the index in the position C D, will be the angular distance of the two objects whose images have been brought into contact. This method of observing with the reflecting circle is the same as that employed when the sextant is used. Wherefore the angles measured are affected with the error that may take place in reading the index, when the two mirrors are parallel.

What has been said is derived from the fundamental principle of the construction of all reflecting instruments, viz. that the angles formed by the visual rays of two objects brought into contact with each other, are equal to double the angle B C D, passed over by the index. But when the index was at B, the mirrors were parallel; therefore the angles formed by the visual rays of two objects in contact, are also measured by double the angle of inclination of the surfaces of the two mirrors: thus, whenever these surfaces make an angle with each other equal to B C D, the same objects will still be found in contact in the field of the telescope. From this remark, a second method of observing single angles with the reflecting circle may be derived; this will be to move the index O P of the small mirror, instead of C B, that of the large one. In fact, the index C B being at zero, and the surfaces of the mirrors parallel, if the index of the small mirror be moved, in the direction in which the divisions are numbered, from P to P′ (*fig.* 31,) until it has passed over an arc PP′ equal to the arc B D, (*fig.* 30) which was previously described by the index of the large mirror, the angle formed by the surfaces of the

two mirrors will be the same in the latter case as in the former, with only this difference, that, after having moved the index of the small mirror, this angle, instead of having its opening on the arc P B D, (*fig*. 30,) will have it on the opposite side, O′ P P′, (*fig*. 31 ;) therefore, the direct rays of the star, or other object, the reflected image of which is observed, will strike the surface of the great mirror from the side O′ P P′, (*fig*. 31,) instead of P B D, (*fig*. 30,) and pass between the small mirror and the object glass of the telescope, before arriving at the large mirror. The means of fulfilling this condition is by causing the instrument to make half a revolution, by turning it about the axis O P of the telescope. In this second position, that face of the instrument which was to the left hand will be turned to the right, and those parts which were the most elevated will become the most depressed, and *vice versa*. In order to obtain the angular distance of the objects which have been brought into contact by this second method of observing, the number of degrees and minutes indicated by the index when the mirrors were parallel, must be subtracted from the number of degrees and minutes given by the same index at the end of the observation.

The observations already described have not any advantage over those that are made with a sextant; on the contrary, the radius of the circle being generally less than that of the sextant, the single angles measured with the latter instrument will be susceptible of greater precision than those that are measured with the former. But if multiple angles are observed, and the two methods above described be combined ; that is, if the two indices are moved alternately, and the direct rays of the heavenly body whose reflected image is observed, fall upon the large mirror, in passing alternately between

the object glass of the telescope and the small mirror, or on the contrary side, then the reflecting circle possesses so many advantages over the sextant, that the latter instrument will probably be wholly abandoned at last.

Suppose that it is required to observe the angular distance of two objects, and that, after placing the index of the large mirror upon the point zero of the graduation, the index of the small mirror is moved until the direct image of one of the objects, and the reflected image of the other, coincide in the field of the telescope. After having completed this observation, if the index of the small mirror be regarded as fixed, and that of the large mirror be made to advance in its turn over the arc B D, (*fig.* 31,) equal to the arc P P′, which has been described by the index of the small mirror, it is evident that when this moving index is in the position C D, the surfaces of the two mirrors will be parallel. Suppose that the index of the large mirror be then caused to advance over the arc D E, equal to B D, it will have actually passed over an arc, the number of degrees of which, marked on the graduated limb, will be double of the observed angle; but the angle that the surfaces of the two mirrors then make with each other, will be the same as that of their mutual inclinations at the end of the first observation: this angle, instead of having its opening on the side O′ P P′, will have it on the opposite side P B D. Therefore, in order that the objects may be found again in contact in the field of the telescope, the instrument must be caused to make half a revolution about the axis of the telescope, so that the direct rays of the heavenly body whose reflected image was observed, may fall upon the large mirror from the side P B D. It should be remarked, that it is not necessary to know the point D, to which the index corresponds when the

mirrors are parallel; by bringing the two objects into contact, the arc B E, which is double the observed angular distance, is obtained directly. The angle thus measured will not therefore be affected with the error that may be committed with regard to the point of parallelism of the two mirrors, which may be a fourth or a third of a minute, that is 15'' or 20''. But another advantage is, that an error at the point of graduation which answers to the index of the large mirror, only influences double the observed angle by its whole value; and consequently the error of the single angle will only be half that of the graduation.

When the second observation is finished, a third may be commenced, and the index of the small mirror made to describe an arc double the observed angle: then the two objects are to be brought into contact. In a fourth observation, the index of the large mirror is moved over the same number of degrees, &c. and when the two objects have been made to coincide, the arc indicated by this index will be the quadruple of the observed angle. Observations of the same angle may be repeated at pleasure; and at the end of the sixth observation, for example, the arc marked by the index of the great mirror will be the sextuple of the angle observed: at the termination of the eighth, it will contain eight times the angle, and so in succession. In general, the observed angle will be equal to the arc passed over by the index of the large mirror, divided by the number of observations; and this angle can only be obtained at the end of each even observation. As the error at that point of the graduation which corresponds to the index at the end of the second observation, only influences the measure of the angle by half its value, at the end of the fourth observation it will only influence it by its fourth, at the end of the

sixth by its sixth: the influence of this error will therefore always be diminished in proportion as the number of observations is increased; and it would be possible to multiply them until this influence is nearly insensible. It is from this last property, and that which renders it unnecessary to know the point of graduation which answers to the index when the mirrors are parallel to each other, that *Borda's* reflecting circle derives its principal advantages over other reflecting instruments. Thus it appears that it is solely in the disposition of the component parts of this instrument that the inventor has found remedies for those imperfections that could not be avoided in its construction; and also for the weakness of our organs. It cannot be too much admired by what a simple combination of mind, he has been able to give that precision to angles measured with this circle which the most able artists could never attain in the construction of the largest instruments.

On verifying the Positions of the Mirrors, and of the Axis of the Telescope with respect to the Plane of the Instrument.

THE first verification is that of ascertaining whether the two mirrors are perpendicular to the plane of the instrument. This is done precisely in the same manner as for the mirrors of the sextant.

The telescope O of the circle is fixed to the index of the small mirror, and is held to the supports R and V by two pieces placed near the ends of the telescope, and which enter the grooves of the supports. Each support has a repelling screw intended to vary the distance of the telescope from the plane of the instrument, as it is wished to have more or less of the quicksilvered part of

the small mirror in the field of the telescope. The supports R and V are each divided in the same manner, which serves to place the axis of the telescope in a situation parallel to the plane of the instrument. As different degrees of inclination may be given to the telescope, by means of the repelling screws placed near its extremities, it becomes necessary to know the point of the division on each support with which the proof-lines of the repelling screws correspond, when the axis of the telescope is parallel to the plane of the circle. This is ascertained in the following manner: first, after laying the circle flat upon a table, and bringing the wires of the telescope parallel to the plane of the instrument, two pieces of brass, similar to those represented by *fig*. 36,* are placed on the limb at two points almost diametrically opposite each other. The instrument is next disposed in such a manner that a very distinct object, at least from 12 to 20 feet distant, is seen in the direction of a line passing through the upper part of the aforesaid pieces of brass; then the telescope is placed at the zero of the division on each support, and the index of the small mirror moved until the same object is seen in the field of the telescope. If the object appear exactly at the intersection of the wires, it will be a proof that the axis of the telescope is parallel to the plane of the instrument, and in all observations, the proof lines of the repelling screw of each support must be placed at the corresponding points of each division. But, when the object does not appear exactly at the centre of the wires, by moving the index of the small mirror, it may easily be brought to it by turning the repelling screw nearest the limb; and then the point of the division which an-

* The French call these pieces of brass *viseurs*.

swers to the proof-line of the screw is to be observed. Suppose it were one or two degrees above that of the repelling screw which is nearest the centre of the instrument, then the proof-line of the former must always be placed the same number of degrees above that of the latter: if the same line had been a certain number of degrees below that nearest the centre, it must be so placed in all observations.

When the axis of the telescope has been rendered parallel to the plane of the instrument, the observation is to be made exactly in the same manner as with a sextant.

On ascertaining when the Surfaces of the Large Mirror are parallel.

BORDA, in his work, entitled " *The Description and Use of the Reflecting Circle*", has given the following table, by which the errors arising from a defect of parallelism of the surfaces of the large mirror may be corrected.

This table is calculated from the hypothesis that the inclination of the surfaces of the large mirror is equal to one minute; Borda means by *right observation*, the observation in which the reflected image comes from the right, and by *left observation*, that in which it comes from the left; he calls *cross observations*, the two successive observations, one on the right, and the other on the left, which render unnecessary the preparatory observation of the parallelism of the mirrors.

Observed Angles.	Right Observations.	Left Observations.	Cross Observations.
0	0″	0″	0″
10	2	1	2
20	6	2	4
30	10	1	6
40	16	0	8
45	19	1	9
50	23	2	11
55	28	4	12
60	33	6	14
65	38	8	15
70	47	10	18
75	55	13	21
80	1′ 4	16	24
85	1 15	19	28
90	1 28	23	32
95	1 43	28	37
100	2 1	33	43
105	2 23	38	53
110	2 50	47	1′ 2
115	3 23	55	1 12
120	4 5	1′ 4	1 31
125	5 0	1 15	1 53
130	5 58	1 28	2 15

We shall now explain the preliminary operations which should be performed before this table is used.

After ascertaining that the two mirrors are perpendicular to the plane of the instrument, and that the axis of the telescope is parallel to it, two very distant terrestrial objects are to be chosen, the outlines of which are nevertheless well defined, and their angular distance being not less than 120°. This distance is to be measured by numerous observations: as soon as these observations are finished, the large mirror should be taken from its box, and replaced in a contrary way; that is,

with that edge which was nearest the telescope in the first position of the mirror, the farthest from it in the second; the large mirror is then to be rendered perpendicular to the plane of the instrument, and the angular distance between the two terrestrial objects to be measured again by the same number of observations as before. If the two series of observations give equal measures for this distance, it will be a proof that the surfaces of the mirror are parallel; and then the observed angles will not want any correction. But if these two measures of the distance are not equal, the large mirror is prismatic, and the half of their sum will be free from error; hence it follows that half their difference will be the error of the observed angles. This error should be added to the angle measured when the mirror is in its first position, if that angle is less than the angle given by the second series of observations; but, should it be greater, the error must be subtracted.

Borda's table affords the means of calculating, by a simple proportion, the error of the observed distances, whatever may be the inclination of the surfaces of the large mirror. The proportion is the following; as the error indicated by the table, which answers to the distance observed for verifying the instrument, is to the error given by the same table for any other angle, 90°, for example, so is the error found by the verification, to a fourth term, which will be the error arising from the large mirror when an angle of 90° has been measured.

Whenever the observations are repeated with a circle, the direct rays of the object, seen by reflection, fall upon the surface of the large mirror alternately from the sides P B D, (*fig*. 31,) and O′ P P′; the angle of incidence will sometimes be on the side where the two surfaces of the mirror are nearest to each other, and sometimes on the opposite side. The errors of the observed angles will there-

fore take place in contrary directions; but they will be greater when the rays of the object seen by reflection fall upon the large mirror from the side P B D, than when the same rays pass between the small mirror and the object glass of the telescope; because their obliquity is greater in the former case than in the latter. At the end of each even observation, after the half, the fourth, or the sixth of the observed angles has been taken, the error will be equal to half the difference of these two errors.

It ought to be remarked, that, when the direct rays of the object seen by reflection come from the side P B D, the observations are the same as those that would have been made with a sextant. Therefore, the errors arising from a defect of parallelism in the surfaces of the large mirror will be less with the circle than with the sextant. It has been observed that these errors are the least when the direct rays of the object seen by reflection pass between the small mirror and the object glass of the telescope; it may therefore be regarded as a general rule, that, when it is wished to observe a single angle, the index of the large mirror must be fixed at the point zero of the graduation, and the index of the small mirror moved in the direction of the same graduation.

On the Coloured Glasses, and the Parallelism of their Surfaces.

The coloured glasses of the reflecting circle, (*fig.* 33,) are detached from the body of the instrument: if it be wished to weaken the light of the heavenly body seen by reflection, they are placed at H, (*fig.* 28 and 29,) in the socket made to receive them; but, when the light of the body seen directly is to be weakened, they are placed at K. The two surfaces of these glasses should

be exactly parallel, in order that they may not occasion any error in the measures of the single angles; however, if they are not parallel, the errors of the observed angles, occasioned by this imperfection, are to be corrected in the following manner.

Place the index of the large mirror upon the point zero of the graduation, and that of the small mirror upon the point where the two mirrors are parallel; put a coloured glass in the socket H, and another in that at K; then the sun must be observed, and the edges of his direct and reflected images brought into contact by moving the index of the small mirror. When this first observation is finished, the glass in the socket H must be turned so as to present its other surface to the large mirror; and, if, on observing the sun again, the edges are still found to be in contact, the surfaces of the coloured glass are parallel, and the single angles observed with this glass will not want correction. But, if the disks of the two images are either distant from or cover each other, the repelling screw of the index of the large mirror must be turned, and the images brought again into contact. The arc passed over by the index of the large mirror will be double the error occasioned by the defect of parallelism of the surfaces of the coloured glasses. The same observation may be repeated several times; in the fourth observation, the arc passed over by the index will be quadruple of the error, and in the sixth, it will be sextuple. Generally, the error from the coloured glass will be equal to the arc passed over by the index of the large mirror, divided by the number of observations. The position of the line C N remaining constant with respect to the coloured glass, the correction of that error will be the same for all the observed angles. By operating in the same manner, the errors of all the

coloured glasses, similar to those of *fig*. 33, may be ascertained when they are placed at H. It will be equally easy to ascertain the errors of these same glasses when they are placed at K; the process will be the same, except that, between the observations, the glass in the socket K must be turned instead of that in H. The position of the line O N P is invariable with respect to the situation of the coloured glass placed at K, and the correction found in this case will be applicable to all the angles.

The advantage which, in general, is derived from increasing the number of observations made with a circle, is as great with respect to these errors as to all the others; for, whatever may be the inclination of the surfaces of the coloured glasses, the error of the angle measured will be nothing at the end of each even observation. It has been already shown that the error was the same in all the observations; thus it will have the same value whether the direct rays of the object seen by reflection fall upon the large mirror from the side P B D, (*fig*. 31,) or from the opposite side. But the instrument is reversed in the second observation; consequently, if the refraction experienced by the reflected ray in passing through the coloured glass, tends to elevate this ray in the first observation, the same refraction ought to depress it equally in the second. Then the half of the double angle, or the fourth of the quadruple angle, and so on in succession, will be free from error.

It is to be regretted that the glasses of *fig*. 33 cannot be used in all circumstances. In fact, if, through the centre C of the large mirror and the edges S S of the mounting of one of these glasses, the lines C M, C Y be drawn, whenever the direct rays of the body seen by

reflection fall in the angular space M C Y, they meet, before they arrive at the large mirror, the coloured glass or its mounting, and the observation will be imperfect. In the circles constructed according to the dimensions fixed by *Borda*, the angle M C Y should be 28° 40′. If the line C X be drawn parallel to the axis of the telescope, the angle Y C X will be 5° 20′. Hence, in the case in which the direct rays of the object seen by reflection pass between the small mirror and the object glass of the telescope, it is impossible to make use of the glasses of *fig*. 33, whenever the observed angle is between 5° 20′ and 34°. The glasses of *fig*. 34 must then be employed, and placed before the large mirror in the sockets *q*. *q*. But these glasses may occasion errors rather considerable. In fact, they are first traversed by the rays falling upon the large mirror, and then again by the reflected rays; therefore, the rays of the body seen by reflection experience a double refraction. Besides, the errors arising from the defect of parallelism of their surfaces vary in each observed angle, because the angles that the direct and reflected rays make with these surfaces are not the same. The glasses of *fig*. 34 must, therefore, be used only for observing altitudes between 5° 20′ and 34°. At all events, the obliquity of the rays of the body seen by reflection will not be very great when the angles are below 34°, and the errors occasioned by the defect of this kind of glasses, will be less in the circumstances in which we are obliged to use them than in all others. The distances between the sun and moon are always greater than 40°, and those between the moon and stars are so seldom less than 34°, that the coloured glasses placed before the large mirror are scarcely of any use in observations of distances.

Point of Parallelism of the two Mirrors.

WHEN simple angles are to be measured with the reflecting circle, the point of the division, which corresponds to the index of the small mirror when the surfaces of the two mirrors are parallel, must be known; supposing, however, that the index of the large mirror has previously been brought to the zero; this is what is called the *point of parallelism* of the two mirrors; and the process for finding it is the same as with a sextant; with only this difference, that, instead of taking half the difference of the two arcs observed for obtaining the rectification of the sextant, that is, the point of parallelism of the two mirrors, half the sum of the two arcs, given by the index of the small mirror at the end of each observation, must be taken when the circle is used. The reason is very simple, and arises from the index of the small mirror being on a part of the limb in which the division is continued.

Observations of the Altitudes of the heavenly Bodies, and of the distances between the Moon and the Sun, or the Stars.

THE simple observation of the altitude of any of the heavenly bodies is the same with the circle as with the sextant. The index of the large mirror is first to be fixed at the point zero of the graduation; then, after having placed the index of the small mirror upon the point of parallelism of the two mirrors, the heavenly body, of which the altitude is required, is to be observed directly in the telescope. When the sun is to be observed, a coloured glass is placed behind the small mirror. The reflected image is then to be depressed, still preserving it in the field of the telescope; and lastly, after having re-

moved the second coloured glass, this image is to be brought into contact with the horizon. The arc indicated by the index, diminished by that which answers to the latter, when the surfaces of the mirrors are parallel, is equal to the observed altitude. The index of the small mirror may also be fixed at the point of parallelism, and that of the large one moved; in which case, the arc marked by this index will be the observed altitude. The reasons for which it is preferable to use the index of the small mirror, and to regard that of the large mirror as being fixed, have been explained at the conclusion of the article on the parallelism of the surfaces of the large mirror.

The reflecting circle can possess all its advantages only when the observations are repeated; single altitudes must therefore be observed only in cases in which it is impossible to make several observations in succession, as when the meridian altitude of a heavenly body is required. In all other circumstances the observations should be repeated. From the rising of the heavenly bodies to their passing the meridian, their altitudes increase; and, from the instant of this passage to that of their setting, they decrease. In both cases the motions in altitude are very unequal; nevertheless, the variations which they experience are not so considerable as to prevent us from supposing, without sensible error, that, during the short interval of four or six observations, the changes in altitude are proportional to the time. If several altitudes are to be observed when this supposition is nearest the truth, that is, when the Sun is not too near the meridian, the method of proceeding is the following:—The hour, minute, and second, in which each observations is made, are to be written down; and the arc passed over by the index, divided by the number of observations, will be the mean altitude corresponding to the

mean time of the observations. In fact, the first observed altitude will be that of the body at the instant of the first observation; the second altitude will be equal to the first, plus or minus the quantity which the body has ascended or descended in the interval between the first and second observations, or its actual altitude at the instant of the second observation. The arc, described by the index of the large mirror, will, therefore, be equal to the sum of the two observed altitudes; and, if the motion in altitude was sensibly proportional to the time, half the sum of these two altitudes, or half the arc reckoned on the instrument, will be the mean altitude corresponding to the mean time, or to half the sum of the times of the two observations. When four observations are taken, the arc described by the index will be the sum of the four observed altitudes; and a fourth of this sum will give the mean altitude corresponding to a fourth part of the sum of the times of the four observations, or to their mean time. Should six observations be made, a sixth of the arc passed over by the index must be taken for the mean altitude corresponding to the mean time, and so on in succession for any greater number of observations.

The quantity by which the distances of the moon from the sun or the stars vary in a given interval of time, is much less than the changes in altitude in the same interval; and the supposition, that the changes in the distances are proportional to the time, may be regarded as being very exact in practice. The case in which the moon is very near the meridian, and passes it at a very great altitude, must, however, be excepted. Though the changes in the true distance are not affected by this circumstance, yet, as the apparent distance, which then experiences very great and very unequal changes is observed, this hypothesis may become the cause of errors, the influence of which, on the calculated true dis-

tance, and consequently upon the longitude deduced from it, would be very sensible; for which reason, distances should never be taken during the half hour which precedes, and that which follows the moon's passage over the meridian.

It should always be ascertained before observing distances, if the mirrors are perpendicular to the plane of the instrument. At all events, as the circle is much lighter than the sextant, it is more easily held in all positions; it will really be troublesome only when the heavenly body that is seen directly has a very great altitude. The method of observing the contact, with a circle, in a plane parallel to that of the instrument, is the same as that used with a sextant.

In order to avoid useless trouble in bringing the images of the two heavenly bodies, of which the distance is to be observed, into the field of the telescope at each observation, search beforehand in the Nautical Almanac what this distance ought to be nearly. When this is found, it will be easy to conclude the position that the indexes should have at the instant of each observation.

For instance:—Suppose that the mirrrors of the circle that is to be used are parallel, when the index of the small mirror answers to 471° 30′; and that the distance is 80°, add the single distance of 80° to 471° 30′, and we shall have 551° 30′ for the first position of the index of the small mirror.

Double the distance, or 160°, will be the position of the index of the large mirror in the second observation. The position of the index of the small mirror in the third observation will be 551° 30′ + 160°, or 711° 30′. That of the large mirror in the fourth observation will be 160° + 160° or 320°; and so on, by always adding double the distance to the number which indicates the preceding

position of the same index. The successive positions that the indexes ought to have, may be written down in the following manner:—

Positions of the Indexes.

Observations.	*Large Mirror.*	*Small Mirror.*
1st.......-	....	......551° 30′
2........	......160°	
3........	....	......711 30
4........	320	
5........	....	......151 30
6........	480	
7........	....	......311 30
8........	640	
	&c.	&c.

If the indexes be successively placed in the positions which this specimen indicates, the two bodies will be found in the field of the telescope without any difficulty.

Eight or ten observations may be employed for one calculation; but it is best to use only six. Supposing the greatest error in the graduation to be half a minute, or 30″, the error can never exceed a sixth part of this, or 5″; and most frequently it will not exceed 3″. After having read the arc described by the index at the end of the sixth observation, the index of the large mirror might be brought back to the point zero of the division, and other six observations be commenced for a second calculation; but it is better to regard the termination of the arc that has been read, as the point of departure, and to continue to move the indexes alternately in the direction of the divisions; wherefore other numbers should be written in the preceding specimen, and 12 or 18 different positions indicated. The sum of the last six observed distances will be equal to the arc reckoned to the end of

the observation relative to them, minus the arc reckoned to the end of the observation of the first six. This method is preferable to the other for several reasons; first, the errors of the graduation, by which the distances used in each calculation may be affected, will be different, and there will be no reason to fear the influence of the *maximum* of these errors upon the mean longitude which results from the two calculations. On the other hand, as the indexes had different positions upon the limb of the instrument, if there be any small imperfections in its *centrage*, or in any other essential parts of its construction, the influence of these imperfections will never have its *maximum* upon all the longitudes given by the different calculations.

Distances observed with a reflecting circle will not be affected with the error that may be committed in observing the point of division which answers to the index of the small mirror, when the two mirrors are parallel to each other; they will be free from the errors arising from the defect of parallelism in the surfaces of the coloured glasses; and the error arising from the want of parallelism of the surfaces of the large mirror may, in a great measure, be corrected. The greatest errors that are to be feared are those of graduation, which have been valued at 5″ or 6″, after six observations. If half of this quantity be added for the small unknown imperfections to which it is impossible to apply a remedy, it may be concluded, that the reflecting circle gives the distances of the sun and moon to at least 8″ or 9″ nearly. This small error is combined with the inevitable errors that may be committed in bringing the bodies into contact, and it results from their union that the distances will be obtained within 15″ or 18″ of the truth. Hence, it appears that the reflecting circle gives the angles with a

degree of accuracy that leaves nothing to be desired; and mariners cannot be too much induced to use an instrument that possesses such great advantages over all others.

When it is wished to ascertain the astronomical bearing of a terrestrial object with great precision, the distance of the sun from this object may be taken by six or eight observations; but in common cases a double distance will be sufficient.

The angular distances of terrestrial objects may equally be measured with a circle; but unless these distances are intended for delicate observations, it would be better to use a sextant. Even the most common octants have sufficient accuracy when it is only required to take bearings for hydrographical charts.

Reduction to the Centre of the Station.

The centre of the instrument ought to be placed, as often as possible, at the centre of the station, by which means the calculation of this reduction will be avoided, in which an error of one or two seconds may sometimes be committed if the elements are not determined with a certain precision.

It very frequently happens that the instrument cannot be placed at the centre of the station; and this generally takes place in towers and steeples, as most of them are embarrassed with carpentry, or have their centres occupied by a vertical beam, or because the openings are not conveniently disposed for directing the sights to the objects chosen for the vertices of the triangles, when the instrument is placed at the centre. In this case, the angle wants a correction, which consists of the following:—

The observer should have placed the instrument at the point C, the centre of the station (*fig.* 37); but he has been obliged to place it at O, where he has observed the angle A O B: it is required to apply to this angle the necessary correction, in order to obtain the angle A C B, which should have been observed.

For the purpose of abridging the work, let A C B$=$C, A O B$=$O, O C$=r$, A C$=$D, B C$=$G, B O C$=$Y, and consequently A O C$=$(O + Y).

$$C = A\,I\,B - C\,B\,O = O + O\,A\,C - C\,B\,O.$$

$=O+\dfrac{r \sin.(O+Y)}{D}-\dfrac{r\sin.\ Y.}{G}$ In order that this expression may be in seconds, it must be multiplied by the arc equal to radius, that is, by 57° 17′ 44″. 8* or, which is the same, divided by the sine of one second: thus the correction to be applied to the angle O

is $\dfrac{r \sin.(O+Y)}{D.\sin.\ 1''}-\dfrac{r \sin Y.}{G\sin.1''}$

D is the distance of the object to the right, and G of that to the left. It is sufficient to know them to about $\frac{1}{100}$ part; O C$=r$ is obtained by the exact measure from the centre of the instrument to the centre of the station. For this purpose, a point is marked on the top of the tube of the upper telescope, which should be in a perpendicular to the plane of the limb, supposed to be raised from the centre of the circle. This point is used instead of the centre of the instrument, which the telescope covers.

The angle B O C$=$Y is to be measured with the circle. When the angle A O B has been measured, the

* 57° 17′ 44″. 8 $=$ 206264″. 8; log $=$ 5.3144251. The radius in the centesimal division $=$ 63°. 66197 $=$ 636619″.7; log. $=$ 5.8038801.

upper telescope is directed to the object B on the left,* and the lower one to A on the right; the upper telescope is then moved according to the order of the divisions until it points to C: the space which it will have passed over on the limb, will be the measure of the angle B O C. During this operation, the lower telescope should always remain fixed upon the object on the right; if it does not, it may be brought to it by means of the *drum* screw, and the superior telescope directed again to the centre. This should be repeated two or three times, and the mean of the results be taken.

The distance O C is always too small to allow the plumb line, which is suspended in the axis of the station, to be perceived with the telescope. There should therefore be two remarkable points on the upper part of the superior telescope, one towards each extremity, which may serve to direct it to the plumb line. These two points ought to coincide with the direction of the optic axis of the telescope. For this purpose, the telescope may be mounted with two small *pinnules*, or sights.

The above formula is general, and does not require the construction of any figure; it will be sufficient to pay attention to the signs of the sines of (O + Y) and Y.

* This supposes that the divisions of the limb are numbered from left to right. Should they proceed in a contrary direction, the angle A O C, greater than 180°, is then measured. If this angle be called Y′, we shall have Y = 360° — O — Y′; and the formula will become

$$\frac{r \sin.(O + 360° - O - Y')}{D} = \frac{r \sin.(360° - O - Y')}{G},$$

or $\frac{r \sin.(360° - Y')}{D} = \frac{r \sin.(360°) - (O + Y')}{G}$; but sin. (360° — Y′) = — sin. Y′, and sin. (360°) — (O + Y′) = — sin. (O + Y′); hence the formula becomes $\frac{r \sin.(O + Y')}{D} = \frac{r \sin. Y'}{G}$.

Thus the first term of the reduction will be positive when the angle (O + Y) is less than 180°; and it will become negative when (O + Y) exceeds 180°.

The second term will be negative when the angle Y is less than 180°; and positive when it is greater than 180°.

Application of the Formula to an Example.

Let D = 4510 yards, G = 4730 yds, r = 3.96 yds; O = 33° 58′ 37″. 43; Y = 252° 55′; (O + Y) = 266° 53′ 37″.43.

Specimen of Calculation.

1st Term.	2nd Term.
Log. r = 3.96 yds------0.597695	
Comp. log. sin. 1″------5.314425	
+ 5.912120	−5.912120
Comp. log. D----------6.345824	log. sin. Y −9.901872
Log. sin. (O + Y) − 9.999361	comp. log. G 6.325139
− 180″.84------------2.257305	+ 137″.76-----2.139131
	− 180″.84
	Reduction − 43″.08

Thus 43″.08 must be subtracted from the observed angle at the point O, in order to obtain the angle B C A. It may happen that the correction will be nothing; and this will be the case when the point O is situated in the circumference of a circle passing through the three points C, A, and B; then we shall have sin. (O + Y) : sin. Y : : D : G.

The sign — has been given to the log. sin. of (O + Y) and of Y, because (O + Y) and Y both exceed 180°.

The first part of the reduction will always be of the same sign as the sin. of (O + Y); and the second will be of a contrary sign to the sine of Y.

If the operation were performed with grades or decimal degrees, the calculation should be made in the same manner; but then it would be necessary to take instead of the log. sin. of 1″ sexagesimal, that of 1″ decimal, that is, 4.1961199, the arithmetical complement of which is =5.8038801.

What has been said relative to the reduction to the centre of the station, supposes that O C and the angle B O C can be measured, which cannot be immediately done when the centre C is occupied by a vertical beam. Delambre has given formulæ for finding in that case the angle of direction and the distance from the centre; but the calculation, which is very laborious, may almost always be avoided. An easy construction will give, with sufficient accuracy, the distance from the centre and a point in the direction of that centre through which the telescope may be pointed. We cannot expect to ascertain the elements for the reduction to the centre of the station with rigorous precision; as, for this purpose, it would be necessary to suspend a plumb line from the point to which the telescope was directed, which is sometimes ten, fifteen, or twenty yards above the place where the instrument can be fixed. This is generally impossible, even supposing that the floors and interior carpentry would admit the operation, as the aforesaid point is above the top of the building. It is therefore necessary to determine the place which is perpendicularly below the point to which the telescope was pointed, by ascertaining the centre of the interior of the edifice; and it necessarily supposes great perfection in its construction not to apprehend a difference of three or four inches between the centre and this point.

At all events, this difference is very slight, and appears to be much less than the error of observation, particu-

larly if the signals are at a certain distance; it likewise proves that no great degree of precision is required in this instance, and the following construction will be sufficiently accurate in almost all cases.

Suppose that A B E D (*fig.* 38,) is a rectangular beam which occupies the centre of the station C, and O the centre of the instrument; we can neither measure O C, nor the angle of direction M O C, (M is the object to the left,

From the point F, the middle of the side A B, erect a perpendicular F G, making it $= \frac{1}{2}$ B E $= \frac{1}{2}$ A D $=$ F C. Draw O G and O F; from any point H in O G, draw H K parallel to G F, and make I K $=$ H I. The point K will be in the direction O C.

A plumb-line may therefore be suspended on the point K, and the angle M O K $=$ MOC be taken.

If O K be produced to L, and L G drawn, we shall have O L + L G = O L + L C = O C, the distance from the centre. Then in the above formula, there will be known r and Y, with which the reduction to the centre may be calculated.

This construction, which may be made on a floor, requires only a small string and a piece of chalk; the point O may be determined by a plumb-line before the instrument is placed.

The process will be the same for a square beam.

If the beam be hexagonal, let A B D E F G (*fig.* 39) be its contour, O the centre of the instrument. From the points A and B, with the radius A B, describe the arcs I H and K L, which cut each other in X; draw X Z to the middle of A B; join O X and O Z; from any point N of O X, draw N Q parallel to X Z; make P Q = N P: the point Q will be in the direction O C; the angle M O Q = M O C may then be measured. Pro-

long O Q and A B until they meet in V; join Q X; and you will have O V + V X = O V + V C = O C.

If the beam be octagonal, the quadilateral A B E D (*fig.* 40) may be constructed, and the other part of the operation will be the same as for the rectangular beam.

Lastly, if the beam be cylindrical, let ADBE (*fig.* 41) be its circumference. The angles M O A and M O B, which lines drawn from the same point O as tangents to the cylinder, make with the object to the left, may be taken with the repeating circle; and half the sum of these two angles will give the angle of direction M O C.

In order to obtain O C, let O D be drawn equal to the shortest distance from the point O to the beam; and for D C, the cylinder is to be encompassed with a string which must be measured to have the circumference; then, from the logarithm of the number of parts which it contains, subtract the constant log. 0.79818; the remainder will be the logarithm of the number of parts in D C; 0.79818 is the logarithm of the ratio of the circumference to the radius.

Suppose, for example, that the circumference of the cylinder had been found equal to 46 inches.

$$\begin{array}{ll} \text{Log. } 46 \ldots\ldots\ldots & 1.66276 \\ \text{Constant log.} \ldots\ldots & 0.79818 \\ \hline & 0.86458 = 7.3212 \text{ in.} = DC. \end{array}$$

It may still happen that some obstacle will prevent the centre from being seen from the place where the instrument is fixed; let the observer be at O (*fig.* 42): and suppose that an obstacle prevents him from seeing the centre C; in this case, a point B should be chosen, from which both O and C can be seen; and after taking the angle B O C either with the circle or with any other instrument, B C and B O must be measured; then O C is

to be calculated, as well as the angle B O C, which, being added to the angle BOM taken with the circle, will give the angle of direction M O C. This method will be particularly applicable when we are obliged to observe from a gallery without an edifice.

Such are nearly all the difficulties that can be met with. All cannot be foreseen; different expedients must be devised for different circumstances, which skill and experience will readily discover. When openings suitable to our purposes can be made in steeples and towers, the operations may be performed with more ease and accuracy.

Whenever the instrument cannot be placed at the centre of the station, in order to make an observation, it must be so disposed, if it can be done, that all the objects of which the angular distances are required can be seen at once. There will then be only a single distance from this centre to measure, and only one angle of direction to take; the others will be obtained by the successive additions of this angle to the observed angles. If the angle of direction B O C (*fig*. 43) has been observed, we shall have $AOC = AOB + BOC$; EOC, greater than $180° = AOE + AOB + BOC$, &c.

Reduction to the Horizon.

The observed angles are generally in a plane inclined to the horizon of the observer; in this case, they require to be reduced, so that the triangles may be, as it were, projected on a prolongation of the surface of the sea, or on a plane parallel to it.

The elements of this reduction are the zenith distances of the two signals and the observed angle. The use of the repeating circle, in taking zenith distances, has already been explained.

Being at O (*fig.* 44), the angle G O D, between the two objects G and D, has been observed in a plane inclined to the horizon at O; this angle ought to be reduced to P O Q, formed by the lines O P and O Q, which are the intersections of the horizontal plane of the observer with the planes drawn through O G and O D to the centre of the earth.

If at the point O the perpendicular O Z be erected, and with any radius O Z the arcs $d\,Z$, $g\,Z$, and $g\,d$ be described, we shall have a spherical triangle $Z\,d\,g$, in which the three sides are known, viz. the apparent zenith distances $d\,Z$ and $g\,Z$, which have been observed, and $g\,d =$ the arc subtending the observed angle. Let this angle be denoted by A, and the same angle, when reduced to the horizon, by a; call H and h the heights A d and B g of the two signals above the horizon of the observer,(which are the complements of the zenith distances;) then we shall have, by supposing the radius $= 1$, $\cos. A = \cos. a \cos. H \cos. h - \sin. H \sin. h$; or

$$\cos. a = \frac{\cos. A + \sin. H \sin. h}{\cos. H \cos. h},$$ from which we obtain

$$\sin. \tfrac{1}{2} a = \sqrt{\left(\frac{\sin. \frac{1}{2}(A + H - h) \sin. \frac{1}{2}(A - H + h)}{\cos. H \cos. h}\right)}.$$

These two formulæ give the reduced angle, which is always considerable, and indispensably requires to be calculated with great accuracy; but it is generally preferable to ascertain the reduction, which is always very small.

Let $A + x = a$, we shall have $\cos, A = \cos. A \cos. x \cos. H \cos. h - \sin. A \sin. x \cos. H \cos. h + \sin. H \sin. h$; from which

$$\sin. x - \cot. A \cos. x = \frac{\sin. H \sin h - \cos A}{\sin. A \cos. H \cos. h},$$

and $\sin. x + 2 \sin.^2 \frac{1}{2} x \cot. A =$ ----------------

$$\frac{\text{tang.} \frac{1}{2} A \sin.^2 \frac{1}{2} (H + h) - \cot. \frac{1}{2} A \sin.^2 \frac{1}{2} (H - h)}{\cos. H \cos. h}.$$

For the sake of abridging the expression, let the numerator of the second member be denoted by n, we shall have

$\sin. x + 2 \sin.^2 \frac{1}{2} x \cot. A = n \sec. H \sec. h$, and $2 \sin.^2 \frac{1}{2} x \cos. \frac{1}{2} x = n \sec. H \sec. h - 2 \sin. \frac{1}{2} x \cot. A$; from which, after squaring, we have $\sin.^4 \frac{1}{2} x - \sin.^2 A (1 + n \cot. A. \sec. H \sec. h) \sin.^2 \frac{1}{2} x = -\frac{1}{4} n^2 \sin.^2 A \sec.^2 H \sec.^2 h$.

Representing this equation for the purpose of abridging it, by $\sin.^4 \frac{1}{2} x - p \sin.^2 \frac{1}{2} x = -\frac{1}{4} q^2$; we shall have

$$\sin.^2 \tfrac{1}{2} x = \tfrac{1}{4} \frac{q^2}{p}\left(1 + \tfrac{1}{4} \frac{q^2}{p^2} + \tfrac{1}{4}.\tfrac{2}{6} \frac{q^4}{p^4} + \&c.\right)$$

and $\sin. \frac{1}{2} x = \dfrac{\frac{1}{2} q}{p^{\frac{1}{2}}} + \dfrac{\frac{1}{16} q^3}{p^{\frac{5}{2}}}$; and

$2 \sin. \frac{1}{2} x = \dfrac{q}{p^{\frac{1}{2}}} + \dfrac{\frac{1}{8} q^3}{p^{\frac{5}{2}}} = \text{chord } x$: but

$x = \text{chord } x + \frac{1}{24} (\text{chord } x)^3 = \dfrac{q}{p^{\frac{1}{2}}} + \dfrac{\frac{1}{8} q^3}{p^{\frac{5}{2}}} + \dfrac{\frac{1}{24} q^3}{p^{\frac{3}{2}}}$,

or $x = \dfrac{q}{p^{\frac{1}{2}}} + \dfrac{\frac{1}{8} q^3}{p^{\frac{5}{2}}} (1 + \frac{1}{3} p.)$

So that by substituting the values of p and q in this expression, developing as far as n^3, and reducing

$$x = n \sec. H \sec. h - \tfrac{1}{2} (n \sec. H \sec h)^2 \frac{\cot. A}{\sin. 1''} +$$

$$+ \tfrac{1}{3} (n \sec. H \sec. h)^3 \left(\frac{\frac{1}{3} + \cot.^2 A}{\sin.^2 1''}\right).$$

The first term will generally be sufficient, and the second always. In all cases we may see how easy it is to unite the last two terms in a table, which must be twice entered.

The quantity $n = \text{tang.}\ \frac{1}{2}A\ \text{sin.}^2\ \frac{1}{2}(H + h) - \text{cot.}\ \frac{1}{2}A\ \text{sin.}^2\ \frac{1}{2}(H - h)$, is calculated by means of two tables of easy use. The first gives for each value of $(H + h)$ and $(H - h)$, for every minute, the quantity $10000\ \text{sin.}^2\ \frac{1}{2}(H \pm h)$; the second gives for each value of A, for every ten minutes, the quantities $0''.0001$ tang. $\frac{1}{2}$ A, and $0''.0001$. cot. $\frac{1}{2}$ A. Table III. gives the factor sec. H sec. h, which in geodesic operation differs very little from unity; the arguments in it are H and h. Table IV. gives the sum of the two following terms; the arguments in it are (n sec. H sec. h), and the angle A. These tables for the reductions will be found at the end of this volume, calculated according to the sexagesimal system.

EXAMPLE.

Let the observed angle A be $61°\ 9'\ 27''.3$ of the old division of the circle, and the apparent zenith distances of the two objects, $91°\ 25'\ 51''$ and $91°\ 32'\ 45''$; let it be remembered also, that the quantities previously denoted by H and h, are the complements of the zenith distances.

Let the two zenith distances, which are the respective complements of H and h, be denoted by D and d. If D and d are both greater than 90°, H and h will be $(D - 90°)$, (and $d - 90°$): $(H + h)$ will be $= (D + d) - 180°$, and $(H - h)$ will be $= (D - d)$.

If D and d are both less than 90°, than H and h will be $(90° - D)$ and $90° - d)$; $(H + h) = 180° - (D + d$, and $(H - h) = (D - d)$.

Lastly, if D is greater and d less than 90°, H will be $= (D - 90°)$ and $h = (90° - d)$; $(H + h)$ becomes $= (D - d)$ and $(H - h) = (D + d) - 180°$.

This premised,

$$\begin{array}{ll}
\phantom{\text{and }}91^\circ\ 25'\ 51'' & \\
\text{and }\underline{91\ \ 32\ \ 45} & H = 1^\circ\ 32'\ 45'' \\
(H + h) = 2^\circ\ 58\ \ 36 & h = 1^\circ\ 25\ \ 51 \\
(H - h) = 6'\ 54 &
\end{array}$$

$$\begin{array}{lrlr}
(H + h)\text{ 1st table} & + 6.746 & (H - h)\text{ 1st. table} & + 0.010 \\
\text{Tang. A 2nd. tab.} & + \underline{12.19} & \text{Cot. A 2nd table} & - \underline{34.89} \\
 & 60714 & & - 0.3489 \\
 & 6746 & & + \underline{82.2337} \\
 & 13492 & n = & + 81''.8848 \\
 & \underline{6746} & & \\
 & + 82.23374 & &
\end{array}$$

The first part of the reduction is always additive, and the second subtractive. In order to obtain this reduction more accurately, multiply it by (sec. H sec. h); table III. with H and h gives for this factor 1.0007.

$$\begin{array}{lr}
n = & 81''.8848 \\
\text{Sec. H sec. } h & \underline{1.\ 0007} \\
n\text{ (sec. H sec. }h) & 81.94216.
\end{array}$$

Lastly, if regard be had to the last two terms of the formula, table IV. must be used, which is calculated upon the supposition, that (n sec. H sec. h) is 100''.

With the observed angle 61° 9', we find in table IV. the correction $= 0''.013$: this is what it would be necessary to subtract if (n sec. H sec. h) were 100''; but, as that quantity is only 82'', the correction $- 0''.013$ must be multiplied by $\left(\frac{82}{100}\right)^2 = 0.6724$; and $-0''.013 \times 0.67 = -0''.009$.

$$\begin{array}{lr}
(n\text{ sec. H sec. }h) & + 81''.94 \\
 & - \underline{0.\ 009} \\
 & + 81''.93 = 1'\ 21''.93 \\
\text{Observed angle} & 61^\circ\ 9'\ 27''.3 \\
\text{Reduction} & \underline{1'\ 21.\ 93} \\
\text{Angle reduced to the horizon} & 61^\circ\ 10'\ 49''\ 23.
\end{array}$$

Thus it appears, that we would have ... we

may always confine ourselves to n when the reduction is a small number of seconds, and the zenith distances do not differ much from 90°.

The following operation gives the solution of the spherical triangle by the exact formula,

$$\sin.\tfrac{1}{2}a=\sqrt{\left(\frac{\sin\tfrac{1}{2}[A+(H-h)]\sin.\tfrac{1}{2}[A-(H-h)]}{\cos.H\cos.h}\right)};$$

by employing the same given quantities as above, so that the two methods may be compared, when it is thought proper.

$$\begin{array}{l} H\text{-----------}1^{\circ}\ 32'\ 46''\\ h\text{-----------}1.\ 25\ 51\\ \hline (H-h)\text{----}0^{\circ}.\ 6'\ 54'' \end{array}$$

$$\begin{array}{rr} A=61^{\circ}\ 9'\ 27''.3 & A=61^{\circ}\ 9'\ 27''.3\\ (H-h)\ 0\ \ 6'\ 54'' & (H-h)\ 0\ \ 6'\ 54\\ \hline A+(H-h)=61^{\circ}\ 16'\ 21''.3 & A-(H-h)=61^{\circ}\ 2'\ 33''.3\\ \tfrac{1}{2}[A+(H-h)]=30^{\circ}38'10''.65 & \tfrac{1}{2}[A-(H-h)]=30^{\circ}31'16.''65 \end{array}$$

$$\begin{array}{l} \text{Comp. log. cos. }H\text{-----}0.0001581\\ \text{Comp. log. cos. }h\text{-----}0.001350\\ \text{Log. sin. }\tfrac{1}{2}[A+(H-h)]\text{--}9.7072179\\ \text{Log. sin. }\tfrac{1}{2}[A-(H-h)]\text{--}9.7057427\\ \hline \text{Log. sin.}^2\ \tfrac{1}{2}a\text{------------}19.4132541\\ \text{Log. sin.. }\tfrac{1}{2}a\text{------------ }9.70662705=30^{\circ}\ 35'\ 24''.59. \end{array}$$

Therefore a, the reduced horizontal angle, $=61^{\circ}\ 10'\ 49''.18$. This is within about 0. 05 of a second of the angle found above. If $\frac{1}{2}$ unity be added to the 8th figure of the log. sin. $\frac{1}{2}a$, it will give exactly $61^{\circ}\ 10'\ 49''.23$; but the log. sines in the common tables are only correct to about $\frac{1}{2}$ a unit in the 7th figure; from which we may conclude, that tables which give more than 7 figures should be used, in order to answer the reduction to 0.01 of a second; an accuracy which the approximating formula gives, and which may be more commodiously obtained by its means, by employing it, however, in the cases

where H and h do not exceed 3° or 4°. If $H = h$, then $H - h = 0$, and the formula $\sin.\ \frac{1}{2}\ a$

$$= \sqrt{\left(\frac{\sin\ \frac{1}{2}[A + (H - h)]\ \sin.\ \frac{1}{2}[A - (H - h)]}{\cos. H. \cos.\ h}\right)},$$

becomes $\sin.\ \frac{1}{2}\ a = \frac{\sin. \frac{1}{2} A}{\cos. H}$; n, in the approximating formula, becomes $\text{tang.}\ \frac{1}{2}\ A\ \sin.^2.\ \frac{1}{2}\ (H + h)$, and the correction will only have a single part.

Correction for the Eccentricity of the lower Telescope.

In the repeating circles, the axis of the inferior telescope does not pass through the centre of the instrument; this construction causes the observed angle to require a correction.

On beginning to observe an angle A C B (*fig.* 45,) the upper telescope is brought to the object A (*), to the right, in the direction C A. If the lower telescope were concentric and brought into the direction C B, the intercepted arc of the limb would give the angle sought; but because of the eccentricity C D, the lower telescope, which is fixed at D, takes the direction D B.

Therefore, when the lower telescope is next directed to the object A, the point D, by the motion of the instrument on its pivot, is transferred to E, and the telescope itself takes the direction A E; so that the motion imparted to the instrument is equal to the angle D C E, and not to the angle A C B.

Now, D C E = A C E – A C D = A C E – (B C D – B C A) = A C E – B C D + B C A = $(90^\circ - A) - (90^\circ$

---

* This supposes that the divisions of the instrument are numbered from left to right.

$- B) + BCA = 90^\circ - A - 90^\circ + B + BCA = BCA$
$+ B - A = BCA + \frac{CD}{CB} - \frac{CE}{CA};$

for, the angles A and B being very small, their sines may be substituted for the arcs; therefore, the upper telescope is removed to the right, beyond the angle A C B, by a quantity $= BCA + \frac{CD}{CB} - \frac{CE}{CA}$.

Hence, in order to bring it to B, it must be made to describe $ACB + \frac{CD}{CB} - \frac{CE}{CA} + ACB = 2ACB + \frac{CD}{CB} - \frac{CE}{CA}$.

Consequently, when we take half the arc measured on the limb, we obtain $ACB + \frac{CD}{2CB} - \frac{CE}{2CA} = \frac{1}{2}$ (arc measured); hence $ACB = \frac{1}{2}$(arc measured) $+ \frac{CE}{2CA} - \frac{CD}{2CB}$; and therefore, in order to have the angle A C B, to half the measured angle must be added $\frac{\frac{1}{2}\text{ eccentricity}}{D} - \frac{\frac{1}{2}\text{ eccentricity}}{G}$; D and G are the respective distances of the objects on the right and left.

In the figure, the eccentricity is to the right; if it had been to the left, it would have been negative, and the correction would have had contrary signs.

In general, the correction is equal to half the eccentricity, reduced into seconds, and divided by the distance of the object which is on the same side with it, minus half the eccentricity, divided by the distance of the object which is on the other side with respect to the eccentric telescope.

The eccentricity varies according to the dimensions of the circles and the diameter of the telescopes; it would be easy to form a table for this correction, when the eccentricity of the instrument has been measured. In the construction of this table, the eccentricity must be expressed in the same unit as is used for the sides of the triangles.

Suppose the sides of the triangles to be calculated in fathoms, and that the eccentricity is $1\frac{1}{2}$ inch; $\frac{1}{2}$ the eccentricity $= \frac{3}{4}$ of an inch, and $\frac{\frac{3}{4}\text{ inch}}{1\text{ fathom}} = \frac{1}{96}$. If the angles are measured in sexagesimal degrees, we shall have $\frac{57^\circ\ 17'\ 44''.8}{96} = 2148''.6$.

Thus, the formula will be $\frac{2148''.6}{D} - \frac{2148''.6}{G}$, if the eccentricity is to the right. It would be easy to calculate the correction for different distances, which might be made to increase for every 1000 fathoms or other measures. Then, with the distance to the right, an additive correction is to be taken from the table, and a subtractive one with the distance to the left; the difference of these two corrections will be the quantity which must be added to, or subtracted from the observed angle for the eccentricity. If the eccentricity were on the left, the additive correction must be taken with the distance to the left, and the subtractive correction with the distance to the right.

But, suppose that the sides of the triangles are expressed in metres, and the angles in grades or decimal degrees; then, let the eccentricity $= 2$ centimetres, $\frac{1}{2}$ the eccentricity $= 1$ centimetre, and $\frac{0^m.01}{1^m} = 0.01$ metre; the radius $= 63^\circ.66197$, and $636619''.7 \times 0.01 = 6366''.2$.

Thus the formula becomes $\frac{6366''.2}{D} - \frac{6366''.2}{G}$ for the eccentricity to the right. A table might also be calculated of this for every 1000 metres.

Lastly, if, in preserving the sexagesimal degrees for the measure of the angles, the sides of the triangles were expressed in metres, we should have, still supposing the eccentricity $= 2$ centimetres, $57^\circ\ 17'\ 44''.8$ or $206264''.8 \times 0.01 = 2062''.65$; and the formula would be $\frac{2062''.65}{D} - \frac{2062''.65}{G}$ for the eccentricity to the right. If $D = G$, the correction will be nothing. Therefore, the effect of the eccentricity on the three angles of a triangle is reduced to nothing.

Reduction to the Axis of the observed Signal.

Signals of which the points are not well defined ought, as much as possible, to be avoided. Such are round, square, and rectangular towers, &c. which are not surmounted with spires. The telescope can only be directed to the estimated axis of these signals, and the error of the observation may come to several seconds; they are also inconvenient from being frequently illuminated obliquely by the sun: in that case, when we think we are pointing to the middle of the signal, we point only to the middle of the enlightened part.

For example, let a b c d (*fig.* 46) be the signal. If, on account of the distance, we could see only the illuminated face a b, the point A has been observed instead of the point M, the centre of the signal; and the observed angle wants a correction equal to the angle A O M; nay, it is not strictly true, that by directing the telescope to

the middle of the face A B (*fig.* 47,) we are pointing it to E. In fact, the line A B, seen from the point O, appears to the eye to agree with the line A C, perpendicular to O F, which bisects the angle A O B ; thus the visual ray is directed to the point F, the middle of A C, and consequently to the point D, and not to the point E, as it ought to be : the error in the observed angle is equal to the angle D O E.

Call A B, F ; A D O, a ; A O = O E, D ; A O B, b ; F E is parallel to O B ; that D F E = D O B = $\frac{1}{2}b$: because the triangle A B C may be considered as right angled, F E = $\frac{1}{2}$ F cos. a.

We have D : $\frac{1}{2}$ F cos. a : : $\frac{1}{2}$ b : $x = \dfrac{b \text{ F cos. } a}{4 \text{ D}}$; the angle $b = \dfrac{\text{A C}}{\text{A O sin. } 1''} = \dfrac{\text{A B sin. } a}{\text{A O sin. } 1''}$; therefore $x = \dfrac{\text{F}^2 \text{. sin. } a \text{. cos. } a}{4 \text{ D}^2 \text{ R sin. } 1''}$.

If the second object is to the left of the observer, the correction will be additive whenever the angle A D F is less than 90° ; at 90° the correction is nothing ; and when the angle exceeds 90°, it will be subtractive. If the second object is on the right, the correction will be contrary to the preceding.

It follows from this, that unless the visual ray be perpendicular to one of the faces of a square, hexagonal, or octagonal signal, it will not be directed to the axis of the signal ; indeed, the correction which gives only one or two hundredths of a second may always be neglected. Let us return to that which depends on the sun.

If from the point O (*fig.* 46,) only the face a b can be seen, the correction is equal to the angle A O M ; now, A O M $= \dfrac{\text{A M sin. A M O}}{\text{A O sin. } 1''}$: if the face b c only could

be seen, the correction would be $B O M = \frac{B M \sin. B M O}{B O \sin. 1''}$.

If the signal be a round tower, the correction will be a little more troublesome to calculate. The following is the method:—

An observer placed at O (*fig.* 48,) at a convenient distance from the tower A D B S, can see that tower only when it is illuminated by the sun, and even then he sees only the enlightened part; and if he take the middle of this part for the axis of the tower, he commits an error, the quantity of which it is required to determine.

Let C S be the direction of the sun at the time of observation, M S will be the azimuth reckoned from the south: make $M S = z$, the semi-circle A S B will be illuminated. Let O C now be the visual ray of the observer; make $M C O = M Q = x$, and draw D E perpendicular to O C.

The semi-circle D A Q E is that which faces the observer; but as the part A D is not illuminated, the part A Q S M E only can be seen: let fall A F perpendicular to D E, E F will be the orthographic projection of the visible arc, and this arc consequently appears like the right line F E, which is less than the diameter by the quantity $D F = C D\, 2 \sin^2. \frac{1}{2} A D = 2 C D \sin^2. \frac{1}{2} Q S = 2 C D \sin^2. \frac{1}{2} (M Q - M S) = 2 C D \sin.^2 \frac{1}{2} (x - z)$.

The error of observation will therefore be exhibited by $C D \sin.^2 \frac{1}{2} (x - z)$. In order to express this quantity in seconds, it must be divided by $O C \sin. 1''$; thus, making $O C = D$, $C D = r$, we shall have $\frac{r \sin.^2 \frac{1}{2} (x - z)}{D \sin. 1''}$ for the required correction.

If the sun and the object to which we compare the tower are on the same side, the correction is additive; but if they are on different sides it is subtractive.

If x is greater than z, the sun will be on the right of the observer. Knowing the latitude of the place and the time of observation, x and z may be calculated; but, as great precision in this case is not required, it will be sufficient to remark, that $(x-z)=QS$, is the supplement of the azimuth observed at the point O, between the sun and the signal.

If the sun can be seen from the point O, the instrument is to be placed in a vertical position; the upper telescope is to be directed to the signal, and the number of degrees observed which the index marks on the azimuthal circle: the same telescope is then to be brought towards the sun, by turning the whole instrument on its column according to the order of the divisions. The arc passed over on the azimuthal circle will be the angle COP, the supplement of QCS. If this angle is less than 180°, the sun will be to the left of the observer; if it be greater, the sun will be on his right.*

Since the sun has advanced during the time of taking the angle, it will be more accurate to have the azimuth for the middle of the observation; for this purpose, an azimuth may be taken before commencing the observation, and another after it is finished: half the sum of the two will give the azimuth for the middle; this azimuth will then agree with the measured angle, which, on account of the motion of the sun during the time of the observation, has increased or diminished in an equidifferent progression.

EXAMPLE.

Let an azimuth taken before the observation be.................................... 217° 15′

* This supposes that the divisions on the azimuthal circle are reckoned from left to right.

brought over..........	217° 15′
Another taken immediately after the observation	208 5
	425° 20′
Azimuth for the middle of the observation...	212° 40′
$(x-z)$......	— 180°
	32° 40′

Also, let $r=2.64$ yards

$D=10728$ yards

$$\begin{aligned}
\text{Log. } r \ldots\ldots\ldots\ldots &= 0.42160 \\
\text{Log. sin}^2.\ \tfrac{1}{2}(x-z) &= \\
\text{Log. sin}^2.\ 16^\circ 20' &= 18.89810 \\
\text{Comp. log. D} \ldots &= 5.96948 \\
\text{Comp. log. sin. } 1'' &= 5.31442 \\
& \overline{30.60360} = 4''.01
\end{aligned}$$

Thus the correction is $4''.01$: it will be additive if the object with which the tower is compared is to the right; and subtractive, if it be to the left.

If the sun could not be seen from the point O (*fig.* 49) a plumb-line may be suspended at A, the shadow of which will fall in the direction A B, and should be sufficiently prolonged to meet C O produced in B; then, after measuring A B, O B, and the angle A O B, the angle A B O, the required azimuth, may be calculated.

Greater precision on this subject would be superfluous. Formulæ might be given for the cases of hexagonal, octagonal towers, &c.; but these investigations are unnecessary, as occasions seldom occur in which it would be required to have recourse to them.

Spherical Excess.

After the observed angles have been reduced to the centre of the station, and to the horizon, and corrected for the eccentricity of the lower telescope, as well as for the

phase of the signal, if necessary, these angles are really the spherical angles comprised between the sides of the triangles, projected upon the surface of the sea, or upon a plane parallel to it.

In this state, the sum of the three angles of a triangle ought to exceed two right angles, by a quantity equal to the surface of the triangle, reduced into seconds, by the known property of the area of spherical triangles; but there is generally a difference of some seconds between the sum of the three observed angles and the sum of the three angles of the spherical triangle; and this difference is properly the error of observation.

It is not long since, in geodesic operations, any regard was paid to the spherical excess of the three angles of a triangle; this excess was confounded with the error of observation, which always was rather great, as the imperfection of instruments did not permit angles to be measured with that accuracy which is now attained. When the sum of the three observed angles was not equal to two right angles, a third of the difference was applied to each angle as a correction; and then the calculation was made as for plane triangles. This agrees with Legendre's theorem, given in the following page.

The triangles formed on the surface of the ground might therefore be treated as spherical triangles in calculating their sides; from the surface of a triangle we should conclude the excess of the three angles above two right angles; then make the three observed angles equal to two right angles plus this excess; and with these new angles the sides of the spherical triangles should be calculated by the known formulæ; but, because triangles on the ground, which are the subjects of consideration in geodesic operations, differ very little from plane triangles, their sides being very small with respect to the ra-

dius of the sphere, it has been suggested that the calculation of the sides might be simplified, and the spherical triangles be brought to plane triangles, the solution of which is less troublesome. This has been done, and two of the methods shall be explained, which may be compared together at pleasure. One of them is an application of a fine theorem of Legendre, the annunciation of which is :—*If, from each angle of a spherical triangle, the sides of which are very small with respect to the radius of the sphere, there be subtracted a third of the excess of the three angles above two right angles, the angles thus diminished may be taken for the angles of a plane triangle, the sides of which are equal in length to those of the proposed spherical triangle.* It is, therefore, necessary to know the excess of the sum of the three angles of the spherical triangle above two right angles. In order to obtain this, the surface of the triangle must be calculated *à priori*, by considering it as a plane triangle; this will give it at least with sufficient accuracy.

If we have two sides, b and c with the contained angle A, the surface will be $S = \frac{1}{2} b\, c \sin. A$. If we have one side a, and the two adjacent angles B and C, the surface will be $S = \frac{1}{2} a^2 \frac{\sin. B \sin. C}{\sin. (B + C)}$. If the spherical excess be denoted by e, we shall then have $e = \frac{s}{r^2}.R$; r being the radius of the earth which should be expressed in the same units as the sides of the triangle; R is the number of seconds comprised in the radius; therefore e will be expressed in seconds.

Thus, in order to have the spherical excess in seconds, to the logarithm of the surface of the triangle must be added (log. R — 2 log. r), which is a constant quantity. If the sides of the triangles are expressed in yards, and

the angles according to the old division of the circle, the log. of r will be $= 6.84264$, and log. $R = 5.31443$, then the (log. $R - 2$ log. $r) = 1.62915 - 10$, which must be added to the logarithm of the surface of the triangle, to obtain the logarithm of the spherical excess in seconds.

But if the sides of the triangles are given in metres, and the centesimal division of the circle is employed for the angles, we shall have the log. of $r = 6.80388$, and log. $R = 5.80388$; from which (log. $R - 2$ log. $r) = 2.19612 - 10$.

Lastly, if the metre be employed for expressing the sides of the triangles, and the sexagesimal division of the circle used for the angles, it will be necessary to add to the logarithm of the surface of the triangle the constant logarithm, $1.70667 - 10$.

Suppose, for example, that we had the two sides a and b expressed in yards, and the contained angle C given.

viz.....$C = 103° \; 19' \; 10''$.

Log. $a =$ 4.917733

log. $b =$ 4.850510

We should have for the calculation of the quantity $\frac{1}{2} ab$ sin. C, which expresses the area of the triangle,

log. a	4.917733
log. b	4.850510
Log. sin. C.....	9.988135
Comp. log. 2....	9.698970
	9.455348
Constant log. —	9.629144
	1.084492 = 12''.15

Thus the spherical excess amounts to 12''.15, because the the sides of the triangle are very great: this excess seldom exceeds 5'', in geodesic operations.

The three observed angles of the triangle ought to exceed two right angles by 12''15; what is wanting will be

the error of observation. The preceding calculation is not necessary for finding the sides of the triangles; it is useful only for making known the error of observation in the angles, and for bringing them to their true value, by applying a third of this error to each angle, in order to render their sum equal to two right angles, plus the spherical excess.

We may satisfy ourselves in calculating triangles, with distributing the excess or the difference between 180° and the three observed angles, by thirds, in order that the sum of the observed angles may become equal to two right angles; unless some particular reason induce the observer to make some other distribution, depending upon the greater or less reliance he has upon the accuracy of any particular angle.

The three angles thus reduced to 180°, are those that are to be employed in calculating the sides of the triangles, considered as plane triangles.

Delambre's method consists in reducing the horizontal angles comprised between the objects, to the angles comprised between the chords which subtend the terrestrial arcs; by this means the spherical triangle will be reduced to a plane triangle, formed by right lines supposed to join the feet of the three signals, projected on the surface of the sea produced.

For the purpose of bringing the spherical angle comprised between two terrestrial objects, to the angle of the chords, it may be supposed that it is required to bring the angle *reduced* to the horizon to another plane, the situation of which is determined by the length of the arcs which contain the angle. In fact, these arcs are the measures of the angles of depression between the plane of the horizontal angles and the plane of the chords; for, the angle comprised between a chord and its tangent,

is measured by half the arc of that chord: hence, this problem is the reverse of that for the reduction to the horizon. The same formula may therefore be used, by changing the signs of the tables.

The arcs of the great circles, including the angle, must be expressed in minutes; for this conversion the following formula may be employed,

$$K\left(\frac{1-\frac{1}{2}e^2\sin^2.L}{R\sin.1'.}\right),$$

in which K is a side of the triangle expressed in known measures; e is the ellipticity of the earth; L the latitude; R the equatorial radius in the same measures as K. This quantity is the angle formed by the normal drawn from one extremity of the arc, and the right line drawn from the other extremity to the point where the normal meets the axis. This angle differs very little from the terrestrial arc. For the regions about 45° degrees of latitude, the following formula may be used;

$$K\left(\frac{1-\frac{1}{4}e^2}{R\sin.1'}\right).$$

When the two terrestrial arcs have been determined, their halves are to be taken, and called P and Q; the sum of these halves will be (P + Q), and the difference (P—Q). With these numbers the two factors are to be sought in the first table, with the angle reduced to the horizon. The tangent and cotangent numbers may be sought in Table II. The sign — is to be given to the tangent, and the sign + to the cotangent.

EXAMPLE.

Let it be proposed to reduce the horizontal angle 61° 10′ 49″.3 comprised between the two sides of 18687 and 22271 English yards, in the latitude 49° 20′, to the

angle of the chords. These sides being respectively denoted by k and K, the corresponding arcs may be expressed in minutes by the formula K $\left(\frac{1\ \frac{1}{2}e^2 \sin^2 . L}{R \sin . 1'}\right)$.

The factor $\frac{1-\frac{1}{2}e^2 \sin^2 . L}{R \sin . 1'}$ must first be calculated, and then may be denoted by M.

Log. $\sin^2$.L = 19.7599268
log. e^2 .... = —3.7766329
log.0.5 ..... = —1.6989700

Log. $\frac{1}{2}e^2 \sin^2$. L = —3.2355297 = number 0.00172
$1-\frac{1}{2}e^2 . \sin^2 . L = 0.99828$

Log. ($1-\frac{1}{2}$ e^2 $\sin^2$. L) = —1.9992524
Comp. log. sin. 1' = 3.5362793
Comp. log. R in Eng. yds. = 3.1567060
6.6922377 = log. M.

Then we shall have

Log. M ....6.6922377 Log. M 6.6922377
log. k = 18687 .. 4.2715396 log. K = 22271..4.3477397
0.9637773 = 9'.20 1.0399774
= 10'.96

Thus P = 10'.96 and Q = 9'.20

10'.96	10'.96
9'.20	9'.20
(P + Q) = 20'.16	(P — Q) = 1'.76
(P + Q) tab. 1st....0.085	(P — Q) tab. 1st....0.000
with the obs.ang. tab. 2. } —12.19	with the obs.ang. tab. 2. } +34.89
6095	0.000
9752	
—1''.03615	

obs.ang. 61° 10' 49''.3
61° 10' 48''.26

The angle comprehended by the chords is therefore 61° 10' 48''.26.

The same reduction must be made for each of the angles, and their sum rendered equal to two right angles, by adding to, or subtracting from each a third of the difference between that sum and two right angles: after which the calculation may be made as for a plane triangle.

Observations of Latitude.

MERIDIAN altitudes either of the sun or of the stars may be employed in determining the latitudes of places on the earth; the method is the same for both. The moment the star passes the meridian is ascertained by means of its right ascension, and several altitudes of it are observed before and after the passage, taking the exact time of the observations by a seconds pendulum.

The invention of the repeating circle enables us to obtain the latitude, in one night, within a second of the truth; an accuracy which before required the successive observations of several years.

In fact, we may take 30 or 40 zenith distances of the same star before and after it has passed the meridian; similar observations may be made on several stars during the same night, and consequently, in a few hours, we may obtain a hundred observations of latitude. The altitudes of the same star are taken by prolonging the series of observations, without reading each of them: it will be sufficient to read the last, and to divide the arc passed over by the number of angles observed, because the motion of the star may be considered as uniform during those small intervals of time.

The different heights of a star taken at some distance from the meridian want a correction to reduce them to the meridian.

Let H Z P R (*fig.* 50) be the meridian, Z the zenith, E the star, P E its distance from the pole, Z E the observed zenith distance; make P e = P E; it is required to determine Z e from Z E which was observed.

Let the latitude of the place, which it is sufficient to know nearly, be denoted by L; the declination of the star by D, and its distance from the meridian, or the horary angle, Z P E, by P; Z e = Z P — P E = (90° — L) — (90° — D) = D — L. It is evident that Z e is less than Z E. Let x be the difference, then Z E = Z e + x = (D — L + x); the spherical triangle Z P E gives cos. Z E = cos. P E cos. P Z + sin. P E sin. P Z cos. P or cos. (D — L + x) = sin. D sin. L + cos. D cos. L cos. P.

It was from this equation that Delambre found x, which he has expressed in the following series, viz.

$$x = \left(\frac{2\,\sin.^2 \tfrac{1}{2}\,P \cos.\,D \cos.\,L}{\sin.\,(D - L) \sin.\,1''}\right) -$$

$$\tfrac{1}{2}\left(\frac{2\,\sin.^2 \tfrac{1}{2}\,P \cos.\,D \cos.\,L}{\sin.\,(D - L) \sin.\,1''}\right)^2 \cot.\,(D - L)\ \sin.\,1'' -$$

$$\tfrac{1}{2}\left(\frac{2\,\sin.^2 \tfrac{1}{2}\,P. \cos.\,D \cos.\,L}{\sin.\,(D - L) \sin.\,1''}\right)^3 \cot.^2\,(D - L) \sin.^2\,1''$$

The third term is always insensible; the second is easily calculated by means of the first, but the first is generally sufficient.

The value of x is to be subtracted from the observed distance when the star passes between the pole and the zenith, according to the figure; hence, in this case, the signs of the above value of x must be changed.

When the star passes below the pole, the signs of x in the preceding value are not to be changed, but (D + L) is used for (D — L).

Lastly, if the stars pass to the south of the zenith, the

signs in the value of x must be changed, and $(L - D)$ written for $(D - L.)$

If the declination of the star be north, D will have its sign changed.

By taking only the first term of the formula, we shall have $2 \sin.^2 \frac{1}{2} P = \sin. \text{ vers. } P$, and $(\text{tang. } D \mp \text{tang. } L) = \frac{\sin. (D + L)}{\cos. D \cos. L}$; this may be put under this form

$$x = \mp \frac{\sin. \text{ vers. } P}{(\text{tang. } D \mp \text{tang. } L) \sin. 1''}.$$

This formula would serve for the calculation of general tables, with the argument $(\text{tang. } D \mp \text{tang. } L)$.

Otherwise, the calculation of tables from the preceding formula is very easy. It is to be remarked that the first term has only the $\sin.^2 \frac{1}{2} P$ variable, and the second only $\sin.^4 \frac{1}{2} P$. The logarithms of the two consecutive numbers of the tables vary therefore only on account of the variation of $\log. \sin.^2 \frac{1}{2} P$ and of $\log. \sin.^4 \frac{1}{2} P$. Thus, when we have the logarithm of the first number of the table, we shall have those of all the others, by adding successively the differences of the logarithms of the $\sin.^2 \frac{1}{2} P$ and $\sin.^4 \frac{1}{2} P$.

It will be sufficient to calculate the second term for every minute of time; the intermediate terms may be concluded from these by an easy interpolation.

Tables of reduction might therefore be calculated for the star or stars which have been chosen for the observation of latitude.

Suppose, for example, that a table of reduction was required for the polar star, which appears to be preferable to all others for observations of latitude.

The latitude of the place being $51° 2' 10'' = L$, and the declination of the star $88° 12' 50'' = D$, we have $(D - L) = 37° 10' 40''$. $(D + L) = 139° 15' 0''$.

Superior passage.		*Inferior passage.*	
log. 2.............	0.30103		
Com. log. sin. 1″ ..	5.31443		
log. cos. D.........	8.49372		
log. cos. L.........	9.79853		
	3.90771	....	3.90771
Com.log.sin.(D−L)	0.21875	Com.log.sin.(D+L)	0.18525
log. *a*........	− 4.12646	log. *a*.........	+ 4.09296
2 log. *a*.......	+ 8.25292	2 log. *a*	8.18592
log. $\frac{1}{2}$	− 1.69897	....	− 1.69897
sin. 1″	4.68557	....	4.68557
cot. (D − L)	0.12008	cot. (D + L)..	− 0.06467
log. *b*..........	+ 2.75754	log. *b*........	+ 2.63513

Thus, the constant logarithms of the first term of the formula are − 4.12646 for the superior passage, and + 4.09296 for the inferior; these logarithms are to be added to the logarithmic differences of $\sin.^2 \frac{1}{2}$ P, to obtain the logarithms of the corrections answering to the different horary angles P, due to the first term.

The constant logarithms of the second term are + 2.75754 for the superior passage, and + 2.63513 for the inferior; these logarithms must be added to the logarithmic differences of $\sin.^4 \frac{1}{2}$ P, in order to have the logarithms of the corrections corresponding to the different horary angles P, of the second term.

For the superior passage, the first term is negative, and the second positive; but, as this last is always much the least, the reduction is subtractive.

For the inferior passage, both the first and second terms are positive, because (D + L) is always greater than 90°.

During the observation of the same star, D varies by reason of the precession, aberration, and nutation; but this variation is so small, that the declination may be

regarded as constant during the interval, though it may be three or four months; but after some years, the tables require recalculating.

When we wish to observe the same star, it will be convenient to make a table, which gives, for the whole interval, the apparent position of the star; that is, its right ascension in time and its distance from the pole nearly, both affected with the precession, aberration, and nutation.

The instants of the observations compared with the time of passage over the meridian will give the horary angles P, with which the proper reductions may be taken from the table. The sum of all these reductions, divided by the number of the observations, will be the mean reduction, which is to be subtracted (for the superior passage) from the mean of all the observed distances, (which is the arc passed over divided by the number of observations:) the remainder will be the apparent distance, as it would have been observed on the meridian.

The method of calculating particular tables of reduction for any star whatever, has been applied to the polar star, and explained; but, instead of these particular tables, it is advantageous, in certain cases, to employ those of a general nature, as when we propose to observe a great number of stars, and to limit ourselves to about a hundred observations for each. The method of constructing these tables is the following:—

The formula found above may be put under this form:

$$x = -\frac{2 \sin.^2 \frac{1}{2} P \cos. D \cos. L}{\sin. 1'' \sin. (L - D)} + \ldots\ldots$$

$$\left(\frac{2 \sin.^2 \frac{1}{2} P \cos. D \cos. L}{\sin. 1'' \sin. (L - D)}\right)^2 \cdot \frac{\sin. 1''}{2 \operatorname{tang}. (L - D)} - \&c.$$

There is not any thing variable in this formula but the factors $\left(\frac{2 \sin.^2 \frac{1}{2} P}{\sin. 1''}\right)$ and $\frac{2 \sin.^4 \frac{1}{2} P}{\sin. 1''}$.

All the rest may be supposed constant during any series of observations, at least, when the body upon which they are made is not the moon. Abstraction may therefore be made from the first term of the formula, of the factor $\frac{\cos. D \cos. L}{\sin. (L-D)}$, or it may be supposed equal to unity, considering only the variable quantity $\frac{2 \sin.^2 \frac{1}{2} P}{\sin. 1''}$ and constructing a table from it, depending solely on the horary angle.

The different values of the variable quantity may be taken from this table for each of the observations; and then the sum of these values being multiplied by the common factor $\frac{\cos. D \cos. L}{\sin. (L-D)} = \frac{1}{\text{tang. } L - \text{tang. } D}$ we shall have the whole correction, such as it would have been obtained from a particular table which gives at once the whole term $\frac{2 \sin.^2 \frac{1}{2} P \cos. D \cos. L}{\sin. (L-D) \sin. 1''}$.

The second term of the formula may be written thus, $\left(\frac{2 \sin.^4 \frac{1}{2} P}{\sin. 1''}\right) \left(\frac{\cos. L \cos. D}{\sin. (L-D)}\right)^2 \cot. (L-D)$, where $\frac{2 \sin.^4 \frac{1}{2} P}{\sin. 1''}$ is the only variable factor, for which a table may be constructed depending on the horary angle. In this table the different values of the variable quantity may be taken for each observation; and the sum of these values, multiplied by the common factor $\left(\frac{\cos. L \cos. D}{\sin. (L-D)}\right)^2 \cot. (L-D)$ will give the whole correc-

tion, relative to the term $\left(\frac{2 \sin.^4 \frac{1}{2} P}{\sin. 1''}\right) \left(\frac{\cos. L \cos. D}{\sin. (L - D)}\right)^2$ cot. (L—D).

We shall thus have the two terms of the value of x. But, in order to facilitate still more the calculation of this value, the two factors $\frac{\cos. D \cos. L}{\sin. (L - D)}$, and $\left(\frac{\cos. L \cos. D}{\sin. L - D)}\right)^2$ cotang. (L — D), might be reduced into tables, for each particular latitude in which they were intended to be used.

The values of $\frac{2 \sin.^2 \frac{1}{2} P}{\sin. 1''}$ must be calculated for every second of the horary angles, at least as far as 16′, so that the quantities may be taken from the table at once, and without any proportional parts: it will be sufficient to calculate, for the same space of time, the values of $\frac{2 \sin.^4 \frac{1}{2} P}{\sin. 1''}$ for every 10″. We may even, very frequently, omit the values contained in this table, also those in that of the factor $\left(\frac{\cos. L \cos. D}{\sin. (L - D)}\right)^2$ cot. (L — D). In fact, in order that the product resulting from these two factors may be sensible, the union of two circumstances is necessary; viz. that the zenith distance may be inconsiderable, and the horary angle rather large: now, when the zenith distance is small, we ought to avoid large horary angles, because the least error in the time of observation would have a very sensible influence on the reduction, and, consequently, on the reduced distance. It will be proper, for example, to discontinue the observations when the reduction increases by $\frac{1}{2}$ or $\frac{1}{3}$ of a second of a degree for a second of time; and this happens at some minutes distance from the meridian when the star is

very elevated. To find the moment at which the reduction increases $\frac{1}{2}$ or $\frac{1}{3}$ of a second, for one second of time, or in general $\frac{1}{n}$th of a second, we have only to make $\frac{1}{n} = dx = d\left(\frac{2\ \text{sin.}^2\ \frac{1}{2}\ \text{P cos. D cos. L}}{\text{sin.}\ (\text{L} - \text{D})}\right) = \frac{d\ \text{P sin. P cos. L cos. D}}{\text{sin.}\ (\text{L} - \text{D})}$; from which

$$\text{sin. P} = \frac{d\,x}{d\ \text{P}} \cdot \frac{\text{sin. (L} - \text{D)}}{\text{cos. L cos. D}} = \frac{\text{sin. (L} - \text{D)}}{15ndt\ \text{cos. L cos. D}}.$$

If we suppose $n = 1$ and $dt = 1''$, that is, if we investigate the time in which $1''$ of time changes the reduction $1''$ of a degree, we shall have $\text{sin. P} = \frac{\text{sin. (L} - \text{D)}}{15\ \text{cos. L cos. D}}$; and as P is generally a small angle, we may suppose the arc proportional to its sine, and say that P decreases as the fraction $\frac{1}{n}$ decreases. It would be useful to calculate tables from this formula, for different latitudes, in which it might be seen how far the observations may be continued in the different cases.

Delambre has given, in the *Connaissance des Temps*, for the year 12 (1804) two tables for the factors $\frac{2\ \text{sin.}^2\ \frac{1}{2}\ \text{P}}{\text{sin. }1''}$ and $\frac{2\ \text{sin.}^4\ \frac{1}{2}\ \text{P}}{\text{sin. }1''}$, calculated for every second in the first, and for every $10''$ in the second, as far as $16'$ of time. There are also similar tables for the factors $\frac{\text{cos. L cos. D}}{\text{sin. (L} - \text{D)}}$ and $\left(\frac{\text{cos. L cos. D}}{\text{sin. (L} - \text{D)}}\right)^2$ cotang. (L — D) for the latitude $48^\circ\ 51'$, which will serve for the different observatories of Paris. There is also subjoined a table for finding the horary angle for the same latitude, when the reduction varies one second of a degree for each second of time. This table is calculated from the formula $\text{P} = \frac{\text{sin. (L} - \text{D)}}{15\ \text{cos. L cos. D}}$.

From this table it may be seen, for example, that for 30° of north declination, the horary angle is 8′.7; from which we may conclude that, about 4′½ before and after the passage over the meridian, we shall have 0″.5 error in the reduction for an error of 1″ in the time of observation; and that at about 3 the error would be nearly ⅓ of a second. Very little dependence can, therefore, be placed on observations made with this declination when the horary angle exceeds five minutes. Now, at five minutes, the second term is 0″.006 × 9.1, (see the subsequent tables) that is 0″.054, a quantity too small to affect the calculation.

At 20° of declination, the horary angle is 11′.9; therefore, the observations may scarcely be commenced but 6′ or 7 before the passage over the meridian; and, in this case, the second term will be 0″.012 × 3.0 = 0″.036.

Hence, it appears that the second term of the reduction may be neglected as far as 30° of north declination. Beyond that height, the use of Borda's circle is perhaps not sufficiently certain, since 10′ of inclination in the plane produces an error of 2″.53, and 5′ an error of 0″.64.

The stars that are only 20° from the pole may be observed at Paris in both their superior and inferior passages, without the neglecting of the second term ever producing a sensible error as far as 16′ from the meridian; as will shortly be demonstrated.

Such is therefore the use of Table V; it shews the duration that can be given to a series of observations. At 30° degrees of north declination it gives 8′.7, that is, the observations may be continued about 9′, four and a half before the passage, and as long after. At 40° of south declination, we find 30′, that is, the observations

may be commenced 15 minutes before the passage, and end 15′ after it, and so on for other declinations.

We shall now give an example of the Tables I. II. III. and IV. inserted at the end of this volume.

Suppose we had twenty-two horary angles, as in the specimen of calculation following the tables. With these angles we take from Tables I. and III. the forty-four corresponding numbers, and place them in two columns. The sum of the first is, in this example, 4324″.5, to which the sign — is prefixed, except in passages observed below the pole; we also take the sum of the second column, which is, in this instance, 3″.33, and always has the sign + prefixed to it.

The logarithms of the two sums are to be taken and written down separately; below each of these logarithms, the arithmetical complement of the number of observations, which is here 22, is to be written.

Thus far nothing depends either on latitude or declination, the operation is always the same; only the horary angles and the number of observations are considered. To complete the calculation, suppose 48° 41′ the latitude, 33° 24′ the south declination. With 33° 24′ or 33°.4 south declination, search, in Table II, the value of the factor $\frac{\cos. D \cos. L}{\sin. (L - D)}$ calling it F, and which is 0.5544; and in table IV, the value of the factor $\left(\frac{\cos. L \cos. D}{\sin. (L - D)}\right)^2$ cotang. (L — D), which may be denoted by f. This factor is equal to 0.043. The logarithm of F being added to the first calculation, and that of f to the second, the sums are the logarithms of 108″.58 and 0″.0065: thus the reduction will be = — 108″.98 + 0″.0065 = — 108″.97 = — 1′ 48″.97.

The second term in this case is insensible: now the

quantity $+ 3''.33$ is a kind of *maximum;* divided by the number of observations it becomes $0''.1514$; therefore, it may be neglected when f does not exceed unity. In looking over the table, it will be seen that below the pole, $0''.1514\, f$ cannot exceed $0''.067$; even above the pole, from 90° to 69° of declination, the second term never exceeds $0''.2$. Wherefore, it may be neglected on the north side, from the horizon as far as 20° of zenith distance.

Towards the south, as far as 41° of height, that is, as long as the declination is south, the second term cannot much exceed $0''.1$; at 10° of north declination it is $0''.2$ at most; it will even be insensible, if, instead of observing during 32′, the time be reduced to 15′, as prescribed in Table V, or even to 20′; for it should be remarked that the $\frac{3''.33}{22}$ will be reduced to $\frac{2''.18}{20} = 0''.109$, if the two extreme observations be suppressed; to $\frac{1''.29}{18} = \frac{0''.645}{9} = 0''.072$, if the four extremes are suppressed. By continuing this examination it would be seen that the use of Tables II. and IV. will seldom be required: at all events, the operation would not be more difficult, nor much longer.

It would be easy, by means of general tables, the construction of which has already been explained, to calculate particular tables for each star and for different latitudes; but this trouble may be avoided, by means of the general tables, especially when the second term of the reduction can be neglected, which is almost always the case; this method is as easy and simple, as when particular tables are employed.

The correction or reduction which is obtained either

from particular or general tables, being applied to the apparent zenith distance of the observed star, gives the true meridional distance, or what it would have been if the observation had been made when the star was on the meridian.

The mean refraction and the correction due to the height of the barometer and thermometer must then be added; and lastly, the distance of the star from the pole. The sum will be the true distance from the pole to the zenith; the complement of which to 90° will be the required latitude.

For an inferior passage, the reduction is additive, and the polar distance substractive.

Any error committed in the declination equally affects the latitude; in order to avoid this, the two passages of the same star over the meridian ought, as often as possible, to be observed on the same night. If the declination be not accurate, the two latitudes will not be equal: they will differ from each other by double the correction of the declination, and the half sum of the two will be the true latitude.

It is difficult to ascertain the vertical position of the nstrument within 2″ or 3″ when the zenith distances are taken; which causes a diminution in these distances,* which is equal to $2 \sin.^2 \frac{1}{2} I \cot. D$; I is the inclinaion of the circle, and D the zenith distance.

It appears that the error increases as D diminishes. The observations should therefore not be made on stars near the zenith; it is much better to choose them in the vicinity of the pole.

* This supposes the distances to be less than 90°.

For ζ of the Great Bear and of the Goat, at 4° from the zenith, the error would be 3″ for 5′ of inclination.

For the polar star at 37° from the zenith, it would only be 0″.29 for the same inclination.

In northern observations, the inclination increases the latitude; but it diminishes it in those to the south.

Unless a star be very brilliant, or one of the first magnitude, it is almost impossible to observe it at the intersection of the wires; it must, therefore, be observed at some distance from it. It is also very difficult to place the wire exactly in the horizontal position.

The error arising from an observation made at some distance from the vertical wire is similar to that which is produced by the inclination of the circle; it has the same sign when the observation is made on a star to the south, and a contrary sign when to the north. The best way is to observe the star at the same physical point in the telescope, and as near the wire as possible.

Observations of the Azimuth.

In order to lay down a series of triangles, it is necessary to know the azimuth of one of the sides, or its inclination to the meridian; the azimuths of all the other sides may then be found by calculation.

Observations of the azimuth are made by taking the distance between the sun and a terrestrial object. Suppose M R (*fig.* 51) to be the intersection of the meridian with the horizon of the place M, and M S the intersection of the same horizon with a vertical circle passing through the centre of the sun; the angle S M R may be calculated; and if the angle S M D, between the sun and a terrestrial object D be observed, the angle D M R, or

the inclination of the side D M to the meridian of M will be known.

A star may also be employed instead of the sun, but then it will be necessary to have a light for the terrestrial object, which is not so convenient.

Azimuths are generally taken when the sun is near the horizon, and this operation should be performed as often as possible when the sun is rising or setting; thus the mean result is rendered independent of the errors produced by those of the declination of the sun, the latitude of the place, and the pendulum. The repeating circle does not give the simple distance of the luminary from the terrestrial object; it can be obtained only by pairs; but as the motion of the luminary, with respect to the signal, may be considered as uniform during small intervals of time, the arc passed over may be divided by the number of observations; and the mean arc resulting from the division is sensibly the distance of the sun from the signal, for the mean instant of the observations.

For the distance between a terrestrial object and the centre of the sun, without respect to its diameter, the vertical wire of the telescopes should be alternately directed to the east and west limb of the disk of the sun. If, for example, the terrestrial object is to the right of the sun, after the upper telescope has been fixed upon the object, the vertical wire of the lower telescope is to be brought to that limb of the disk which appears to the right in the telescope; and when the lower telescope is next directed to the same object, the upper must be brought to that limb which appears to the left.

The observations may be continued in the same manner, and the final distance will be that between the centre of the sun and the terrestrial object.

The time, or the horary angle of the heavenly body, is one of the essential elements of the calculation; an error of one second in the true time would produce one of several seconds in the azimuth: it is therefore necessary, in order to have the true time with great precision at the moment of the observation, to know well the rate of the pendulum, and to take corresponding altitudes, on the day of observation, which should be taken again on the following day. If the same azimuthal observations be repeated for several successive days, any errors that may have been committed either in the measured distance, or the time, will be attenuated.

Now, let N P Z M (*fig.* 52) be the meridian, N the northern point of the horizon, P the pole, Z the zenith, S the true place of the sun, S′ his apparent place, and G the terrestrial object; the pendulum will give for the instant of the observation, the horary angle $Z\,P\,S = P$. There is also known P S, the complement of the declination of the sun $= C$; and P Z, the complement of the latitude $= H$; there will therefore be given in the triangle Z P S: $\cos\ Z\,S$ or $\cos.\ B = \cos.\ P \sin.\ H \sin.\ C + \cos.\ H \cos.\ C$; this is the true distance of the sun from the zenith; and $\sin.\ P\,Z\,S$ or $\sin.\ Z = \dfrac{\sin.\ P \sin.\ C}{\sin.\ B}$. The angle Z is the azimuth of the sun, reckoned from the north.

Let r be the refraction and p the parallax in altitude; then we shall have $Z\,S' = B' = B - r + p$: in the triangle Z G S′, there will be known Z S′, the apparent distance of the sun from the zenith; Z G, the apparent distance of the terrestrial object from the zenith; and G S′ the observed distance; if S′G be made equal D, and $Z\,G = A$, and if we make $R = \dfrac{A + B' + D}{2} - A$

and $R' = \frac{A + B' + D}{2} - B'$, we shall have $\sin.^2 \frac{1}{2}$ $G Z S'$, or $\sin.^2 \frac{1}{2} Z' = \frac{\sin. R \sin. R'}{\sin. A \sin. B'}$

If the instrument has not been placed at the centre of the station, this angle Z' wants a correction, in which case, the reduction given by the formula (page 71) is to be used; and here it must be remarked, that the correction is always reduced to a single term, because the distance of the heavenly body, which is in the denominator of the other term, may be considered as infinite with respect to the distance from the centre of the station. Thus, if the body is to the left of the terrestrial object, the correction will be reduced to $+ \frac{r \sin. (() + Y)}{D \sin. 1''}$; and if it be to the right of that object, the correction will be $- \frac{r \sin Y}{G \sin. 1''}$

EXAMPLE.

Suppose, that on the 22nd of July, 1798, the distance between the sun and a terrestrial object had been taken.

The barometer stood at 30.2 English inches; and the thermometer at 58½° deg. of Fahrenheit.

The arc passed over by the telescope, divided by the number of observations, gave 67° 19′ 34″.7 = D, for the distance between the object and the centre of the sun.

The mean instant of all those of the observations, given by the pendulum, was 6 *h.* 36 *m.* 44 *s.*; the corresponding altitudes taken in the morning, gave 0 *h.* 3 *m.* 36 *s.* for the true noon: those taken on the following day, gave 0 *h.* 4 *m.* 27 *s.* for the true noon; so that the pendulum had gained 51 seconds in 24 hours.

To reduce the instant of observation to true time, 3 *m.* 36 *s.* must be subtracted from 6 *h.* 36 *m.* 44 *s.*; the re-

mainder 6 *h*. 33 *m*. 8 *s*. will be the time elapsed, according to the pendulum, from the preceding noon to the moment of observation. Now, to find how much the pendulum gained in 6 *h*. 33 *m*. 8 *s*. we have the following proportion, viz. 24 *h*. 0 *m*. 51 *s*. : 51 *s*. :: 6 *h*. 33 *m*. 8 *s*. : 13.9 *s*.; which taken from 6 *h*. 33 *m*. 8 *s*. there remains 6 *h*. 32 *m*. and 54.1 seconds for the true time of the observation.

The declination of the sun on the 22nd of July, at 6 *h*. 32 *m*. 54 *s*. was 19° 57′ 22″.5 north; thus P S = C = 70° 2′ 37″. 5;

The latitude of the place = 42° 8′ 32″; thus H = 47° 51′ 28″;

And the distance of the terrestrial object from the zenith = 90° 21′ 10″;

6 *h*. 32 *m*. 54. 1 *s*. = 3^s 8° 13′ 31″. 5 = the horary angle = P.

If the observation had been made in the morning, this quantity would be the supplement of the horary angle, which would consequently be 2, 21° 46′ 28″.5 = — P, unless we reckon the horary angles from one noon to another, or from 0° to 360°; then we should have P = 8^s 21° 46′ 28″. 5.

SPECIMEN OF CALCULATION.

Log. cos. P. =	— 9.1555425		
log. sin. H. =	9.8701003	log. cos. H =	9.8267052
log. sin. C =	9.9731064	log. cos. C =	9.5331395
— 0.0997124-------	8.9987492	+0.2290048----	9.3598447
	+ 0.2290048		
	— 0.0997124		
Cos. B =	+ 0.1292924	log. cos. B =	9.1115729

Therefore B = 82° 34′ 16″.67 = the true distance of the sun from the zenith. B would be greater than 90° if the cos. of B were negative.

Comp. log. sin. B = 0.0036606
Log. sin. P-------- = 9.9955092
Log. sin. C-------- = 9.9731064
Log. sin. P Z S or sin. Z------ = 9.9722762 therefore
$Z = 69^\circ 44' 40''.4$

This solution does not shew whether Z is greater or less than 90°; for the sine is the same in both cases. It is generally known before of what kind the angle Z is; but the doubt when it does occur is easily removed.

The angle P Z S is 180° at noon, and it decreases as the sun approaches the moment of his setting; the instant may be determined at which that angle is just 90°, by considering the triangle Z P S as right angled at Z; for if, in this triangle, we calculate the horary angle P, we shall know at what instant the angle Z was 90°, and we shall also know whether it was acute or obtuse at the moment of the observation. If the calculation, in our example, be made, we shall find the horary angle P less than $3^{sig.}$ 21°, &c. when the angle Z was 90°: it was therefore less than 90° at the instant of the observation; so that making B = ---------------- 82° 34′ 16″.67

Mean refraction for that distance (Table VI. following) -------------	—	6. 49. 35
Correction for temperature (Table VII. following)*----------------	+	0. 28 . 65
Paral. of the sun July 22nd at 7°.25′ of altitude ------------------	+	0 8 . 31
B′ the apparent altitude of the sun....		82° 28′ 4″.28

* Attention must always be paid to take the correction for the temperature given in Table VII. with a contrary sign to that corresponding to the factor $(x + y + xy)$ expressed in that table, because the *absolute refraction* $= r \pm dr$ (r denoting the mean refraction, and dr its correction) ought always to diminish the number answering to $(x + y + xy)$ in the calulations of the azimuthal angles.

	A . . 90°	21′	10″				
	B′ . . 82	28	4.28				
	D . . 67	19	34.7				
$(A+B'+D)$	$=$ 240	8	48.98				
$\frac{(A+B'+D)}{2}$	$=$ 120	4	24.49		120°	4′	24″.49
A $=$	90	21	10	B′ $=$	82	28	4. 28
R $=$	29°	43′	14″.49	R′ $=$	37°	36′	20″.21

Comp. log. sin. A 0.0000082
Comp. log. sin. B′ 0.0037636
Log. sin. R 9.6952823
Log. sin. R′ 9.7854884
Log. $\sin.^2 \frac{1}{2} Z'$ . . . . $=$ 19.4845425
Log. sin.. $\frac{1}{2} Z'$ . . . . $=$ 9.7422712

Therefore

$\frac{1}{2} Z' = 33° \; 32' \; 0''.06$, and $Z' = 67° \; 4' \; 0''.12$

Thus the distance of the terrestrial object from the centre of the sun, reckoned on the horizon, $= 67° 4' 0''.12$, which it would still be necessary to reduce to the centre of the station if circumstances required it.

Let O (*fig.* 53) be the place of observation, N the north point of the horizon, and M the south point, S the sun, and G the terrestrial object; then as S O N $=$ Z and S O G $=$ Z′, we shall have M O G $=$ the azimuth counted from the south $= 180° - (Z + Z')$ Thus the true azimuth will be $180° - (69° \; 44'.40''.4 + 67° \; 4' \; 0''.12) = 43° \; 11' \; 20''.48$, and G is to the west of M : if $(Z + Z')$ exceeded 180°, G would be to the east of M, and we should then have M O G $= (Z + Z') - 180°$; it would have been the contrary if the azimuth had been observed in the morning. This is for the case in which the object is to the left of the luminary.

If the terrestrial object were G′, we should then have N O G′ $=$ N O S $-$ S O G′ $= (Z - Z')$; this would be the

azimuth counted from the north towards the west. And if we had $Z' > Z$, NOG' would be $=(Z'-Z)=$ the azimuth reckoned from the north towards the east; it would be the contrary for an observation made in the morning. Here the object is supposed to be on the right of the sun.

The elements of the calculations of azimuthal observations would be the same, if a star was employed instead of the sun. It would be necessary to know the declination of the star which had been chosen, corrected for aberration and nutation, and also its right ascension, equally corrected; the comparison of the instant of observation with the passage of the star over the meridian would give the horary angle P. The calculation would be the same as in the preceding example; there would not be the parallax to add to the true height to have the apparent height.

The azimuthal observation, by which the position of all the sides in the chain of triangles is determined, ought to be made nearly in the middle of that chain, and in a place of which the latitude has been accurately determined; azimuths of verification are then to be taken at the extremities of the chain, and particularly in the direction of the meridian of the former place, which ought to agree very nearly with the azimuths from calculation.

Too much attention cannot be paid to azimuthal observations: they require great experience and precision in the observers, who should, therefore, exercise themselves some days previous to the observation.

CHAPTER II.

On Refraction, and the Method of determining it when the Earth is considered as Spherical.

I. The air being continually subject to the action of solar light, and holding in solution substances of a different nature, cannot possess a constant density. A ray of light which traverses the atmosphere obliquely, and which is besides progressively attracted by the lower beds of air, is forced to deviate every instant from the direction it followed the preceding instant; it is this effect which is called *Refraction.*

If from the point A (*fig.* 54) a terrestrial object B be observed, the luminous ray by which its image is transmitted will follow the curve B D A, and the object B will be seen in the direction of the tangent to that curve, that is at B′; from which it follows, that refraction makes objects in general appear more elevated than they are; the angle B′ A B therefore measures the effect of refraction.* The determination of this angle is the subject of the present investigation.

* It is here supposed that the curve of refraction is of a single curvature, and that its plane is vertical; but some observers, and chiefly Delambre, have ascertained, in a certain state of the atmosphere, the existence of a horizontal refraction, the effect of which is, indeed, much less than that of the vertical refraction, since it scarcely amounts to a few seconds. Nevertheless, when both these deviations take place, the curve of refraction has necessarily a *double curvature.*

Let C be the centre of the earth (*fig.* 55) and A and B two signals; if from the point A the point B be observed, it will appear at B′ by the effect of refraction; in the same manner the point A will appear to be at A′ when observed at B.

Making the *apparent zenith distances*,

$$ZAB' = \delta, \ VBA' = \delta',$$

and the angles of refraction

$$BAB' = r, \ ABA', = r';$$

we shall have for the *true* zenith distances,

$$ZAB = \delta + r = D$$

$$VBA = \delta' + r' = D';$$

therefore $ZAB + VBA = \delta + \delta' + r + r'$ ....(1)

Again, since the outward angle of a triangle is equal to the sum of the two inward and opposite angles, we shall have

$$ZAB = C + ABC$$

$$VBA = C + BAC;$$

therefore $ZAB + VBA = 2q^{*} + C = D + D'$......(2)

This result shews that the two true zenith distances exceed two right angles by a quantity precisely equal to the arc of a great terrestrial circle drawn from one signal to the other.

So that from the equations (1) and (2) we conclude

$$\delta + \delta' + r + r' = 2q + C,$$

or else, because r is sensibly equal to r',

$$r = \frac{C}{2} - \tfrac{1}{2}(\delta + \delta' - 2q) \text{ ....(3)}$$

Dividing both sides by C, we shall have

$$\frac{r}{C} = \frac{\tfrac{1}{2}C - \tfrac{1}{2}(\delta + \delta' - 2q)}{C} = n \text{ ....(4)}$$

and finally $r = nC$.

* q here signifies a quadrant, or 90°.

n varies according to the state of the atmosphere; and Delambre has remarked, that, in France, the value of n is about 0.075 in summer, 0.08 in spring and autumn, and that it varies from 0.09 to 0.10 in winter. If n were negative, the refraction would depress objects instead of elevating them, but this very seldom happens.

It follows from what precedes, that

$$Z\,A\,B=\delta+r=q+\tfrac{1}{2}C+\tfrac{1}{2}(\delta-\delta'),$$
$$V\,B\,A=\delta'+r'=q+\tfrac{1}{2}C-\tfrac{1}{2}(\delta-\delta').$$

In all these calculations, it is supposed that the instrument is placed at the summits of the signals, which does not take place in practice.* If, for example, the centre of the circle is at the point a, the true distance from the point B to the zenith is $Z\,a\,B$, and its observed apparent distance $= Z\,a\,B'$; denoting this last by Δ, we shall evidently have $Z\,A\,B' = Z\,a\,B' + A\,B'\,a = \Delta + A\,B'\,a$; it remains then to determine the angle $A\,B'\,a$, which is the error committed.

Let $A\,B' = A\,B = B$, $A\,a = d\,H$, and $A\,B'\,a = d\,\Delta$; the triangle $A\,B'\,a$ will give $\sin. d\,\Delta = \dfrac{d\,H \sin. \Delta}{B}$.

Taking the arc for the sine, and reducing it into seconds n° 9, it will become $d\,\Delta = R''\,\dfrac{dH \sin. \Delta}{B}$ --(5).

When the triangles are calculated according to Delambre's method, (which has been explained,) we obtain the chord of the angle C for a sphere, the radius of which is equal to the distance from the horizon of the sea to the

* It will easily be conceived, that, in order that the refraction may be determined with accuracy for the moment of observation, it is essential that the zenith distances of the points A and B be taken by two observers at the same instant.

centre of the earth; and this chord is less than the distance $a\,B = B$. In order to ascertain the error, we might make $d\,\Delta$ a function of the known chord K, and of the radius of the earth (Delambre's Memoirs, page 92); but it will be sufficient, in all cases, to make use of the preceding formula, substituting in it, however, K for B.

The preceding correction being applied to the two observed zenith distances, we shall have, for those which should have been observed at the tops of the signals,

$$Z\,A\,B' = \Delta + d\,\Delta = \delta$$
$$V\,B\,A' = \Delta' + d\,\Delta' = \delta'.$$

Such are the values to be employed in the formulæ (3) and (4), for estimating the refraction.

On the Difference of Level on the Sphere.

II. Two or more points are said to be on the *same level* when they are situated in the same surface, similar to and concentric with that of a calm sea; and a right line perpendicular to the line of gravitation is called an *horizontal line*. On the hypothesis that the earth is a perfect sphere, all the vertical lines or directions of gravitation pass through its centre; but if we consider the terrestrial globe as a spheroid generated by the revolution of an ellipse about its less axis, which is more probable, the vertical lines are normals to the surface of the spheroid, though they do not all pass through the centre of the earth.

These definitions being well understood, let C be the centre of the globe considered as spherical (*fig.* 56) and A, B, two points unequally distant from that centre.

If A B′ be a *line of level*, or a terrestrial arc, the height $B\,B' = H$ will be the difference of level of the two points

A, B; if, besides, $ZAB = D = \delta + r$ be the true distance from the zenith to the point B, we shall have, by observing that $B'AC = q - \frac{1}{2}C$,

$BAB' = 2q - ZAB - B'AC = 2q - D - q + \frac{1}{2}C = q - D + \frac{1}{2}C$,

$ABB' = AB'C - BAB' = q - \frac{1}{2}C - q + D - \frac{1}{2}C = D - C$;

but the triangle ABB' gives, by making the chord $AB' = K$.

$$H = \frac{K \sin. A}{\sin. B} = \frac{K \sin. (q + \frac{1}{2}C - D)}{\sin. (D - C)} = \frac{K \cos. (\frac{1}{2}C - D)}{\sin. (D - C)}.$$

If this triangle be supposed to be right-angled at B', it is easy to perceive that we shall have, with sufficient accuracy, for the difference of level required,

$$H = K \cot. (\delta + r - \tfrac{1}{2}C) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we shall have also,

$$H = -K \cot. (\delta' + r' - \tfrac{1}{2}C) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now the value of H, expressed in a function of the two apparent zenith distances, may be obtained in the following manner:

From No. 1, we have

$$ZAB = q + \tfrac{1}{2}C + \tfrac{1}{2}(\delta - \delta'),$$
$$VBA = q + \tfrac{1}{2}C - \tfrac{1}{2}(\delta - \delta');$$

and it is evident that

$$BAC = 2q - ZAB = q - \tfrac{1}{2}C + \tfrac{1}{2}(\delta' - \delta)$$
$$B'AC = q - \tfrac{1}{2}C$$
$$BAB' = BAC - B'AC = \tfrac{1}{2}(\delta' - \delta)$$
$$B'BA = 2q - VBA = q - \tfrac{1}{2}C - \tfrac{1}{2}(\delta' - \delta).$$

$$\text{Thus } BB' = H = \frac{K \sin. BAB'}{\sin. ABB'} = \frac{K \sin. \frac{1}{2}(\delta' - \delta)}{\cos. \left(\frac{1}{2}C + \frac{\delta' - \delta}{2}\right)}$$

This formula is exact; but in many cases $\frac{1}{2}$ C may be made$=0$; then because the $\frac{\sin.}{\cos.}=$ tang. we shall evidently have $H=K \text{ tang. } \frac{1}{2}(\delta'-\delta)$..............(3).

When $\delta' > \delta$, H is positive; the contrary takes place when $\delta' < \delta$, δ being the observed distance at the place of known elevation, and δ' at that of which the elevation is required. Delambre has also investigated, by a very elegant calculation, the value of $\frac{1}{2}(\delta'-\delta)$ in a function of H; but the little utility of that value induces us to omit the detail respecting it.

K and r might be eliminated from all the formulæ into which they enter; for it has been shewn that $r=n$ C, and it is easy to shew that $K=2\,\varrho \sin. \frac{1}{2}$ C, ϱ being the radius of the earth corresponding with the middle of the chord K.

The method of determining the elevations of the tops of signals above a common horizon, (that of the sea, for example,) is evident from the preceding investigation; thus by subtracting from them the heights of the signals, we shall have the altitudes of the ground above the level of the sea. The example which we propose to give of this calculation will likewise fix the ideas in this respect. But let us suppose for a moment, that the points B, B′, B″..B^n are unequally elevated above a common horizon, so that h' is the elevation of the point B′ above the level of B; h'' that of B″ above B′; d''' the depression of the point B‴ below the point B″; and so on: then it is evident that we shall have generally

$$H-D=\text{the difference of level,}$$

by taking the altitudes h positive, and the depressions d negative, supposing them to be reckoned from left to right. If the result of this formula be + the point B^n will be above B; and if it be affected with the sign — the point B^n will be below the same level.

From this we may conclude, that if N is the height of the point B above the level of the sea, N + H — D will be the height of any other point above the same level. The method which has been here explained is the same as is used in the practice of levelling.

As the difference of level on the spheroid may be calculated in the same manner as upon the sphere, (Mem. of Delambre, page 104), we shall only make one observation on that subject; which is, that instead of the angle C formed at the centre of the earth considered as spherical, we may substitute another angle C or $\varphi = \frac{R''K}{\varrho} (1 - \frac{1}{2} e^2 \sin.^2 L)$.

III. When the horizon of the sea can be seen from the place where the observation is made, it is easy to conclude immediately the height of the place above that horizon, by means of the observed angle between the horizon and the zenith; the following is the method.

If from the place of observation B (*fig.* 57) a tangent B A be supposed to be drawn to the surface of the sea, the radius of the earth $CA = \varrho$ will evidently be perpendicular to A B. If we suppose, besides, a line of level or a terrestrial arc A B′ intercepted between the point A and the vertical line V B, $BB' = N$ will be the height required. Now, making as before the true zenith distance $VBA = D = \delta + r$, δ being the apparent observed distance, and r the refraction, the right-angled triangle C A B will give, because the angle $CBA = q - C$ and $CA = \varrho$,

$$CB = \frac{\varrho}{\sin.(q - C)} = \frac{\varrho}{\cos.\ C},$$

from which it follows that

$$BB' = N = \varrho \left(\frac{1 - \cos.\ C}{\cos.\ C}; \right)$$

but $1-\cos. C=\sin. C \tan g. \frac{1}{2} C$; therefore

$$N=\varrho \tan g. C \tan g. \tfrac{1}{2} C;$$

likewise $C=q-B=q-(2q-D)=D-q$,

and $D=\delta+r$; therefore

$$N=\varrho \tan g. (\delta+r-q) \tan g. \tfrac{1}{2}(\delta+r-q) \ldots\ldots\ldots (4).$$

When r is unknown, its value may be deduced from the equation $r=n\,C$ (No. 1), but it will be more convenient to transform N into a function of n, as follows:

First, $C=\delta-q$, by neglecting the refraction: thus, without sensible error,

$$r=n\,C=n(\delta-q).$$

Substituting this last value in the equation (4), and remarking that we may make the tang. $m\,x=m$ tang. x, when x is very small and m less than unity or does not much exceed it, we shall have

$$N=\tfrac{1}{2}\varrho \tan g.^2 [\delta-q+n(\delta-q)]=$$

$$\tfrac{1}{2}\varrho \tan g.^2 [(1+n)(\delta-q)].$$

Therefore, very nearly,

$$N=\tfrac{1}{2}\varrho(1+n)^2 \tan g.^2 (\delta-q) \ldots\ldots\ldots\ldots (5).$$

Investigation of the Formulæ for expressing, in a Function of the Latitude, Parts of the Elliptic Meridian of the Earth.

IV. Let C E be the radius of the equator, and P the pole (*fig.* 58). If through the point A the tangent A T be drawn to the elliptic arc P A E, the right line A M, perpendicular to A T, will be the normal at that point, and the angle A L T=F A T will be the latitude of the point A.

The equation of the ellipse is $a^2y^2+b^2x^2=a^2b^2$; and for the point A the co-ordinates of which are x', y', we shall have

$$a^2\,y'^2+b^2\,x'^2=a^2\,b^2.$$

At the same point A, the equation of the normal A L is

$$y-y' = \frac{a^2 y'}{b^2 x'}(x-x')\text{; and if we make } y=o,$$

we shall have the absciss C L, or

$$x = \frac{a^2-b^2}{a^2} x'\text{;}$$

from this it is easily concluded that the normal A L$=n$, has for its value,

$$n = \frac{b}{a}\left[b^2 + \frac{a^2-b^2}{b^2} y'^2\right]^{\frac{1}{2}}.$$

Let A L F$=$L, we shall have $y'=n$ sin. L, and consequently

$$y'^2 = \frac{b^2}{a^2}\left[b^2 + \frac{a^2-b^2}{b^2} y'^2\right]\text{sin.}^2 \text{ L};$$

from which is obtained

$$y'^2 = \frac{b^4 \text{ sin.}^2 \text{ L}}{a^4-(a^2-b^2)\text{ sin.}^2\text{ L}} \cdots\cdots\cdots\cdots(1);$$

and from this

$$n = \frac{b^2}{a}\left[1 - \frac{a^2-b^2}{a^2}\text{sin.}^2\text{L}\right]^{-\frac{1}{2}} \cdots\cdots\cdots\cdots(2).$$

If, for the sake of abridgment, we make, in this result, $a=1$, and $1-b^2=e^2$, where e denotes the ellipticity, we shall have

$$n=(1-e^2)\left[1-e^2\text{ sin.}^2\text{ L}\right]^{-\frac{1}{2}} = \frac{1-e^2}{(1-e^2\text{ sin.}^2\text{ L})^{\frac{1}{2}}} \cdot (3).$$

and the equation (1) becomes

$$\text{A F}=y'= \frac{(1-e^2)\text{ sin. L}}{(1-e^2\text{ sin.}^2\text{ L})^{\frac{1}{2}}} \cdots\cdots\cdots\cdots(4).$$

On the same hypothesis, the equation of the ellipse may be changed into

$$y'^2 = (1-e^2)(1-x'^2)\text{;}$$

hence, by means of the preceding equation,

$$\text{C F} = x' = \frac{\text{cos. L}}{(1-e^2\text{ sin.}^2\text{ L})^{\frac{1}{2}}} \cdots\cdots\cdots\cdots(5).$$

For the same reason, the value of C L, found above may be changed into

$$C\,L = e^2\,x'\,;$$

but x' is given by the equation (5), therefore

$$C\,L = \frac{e^2 \cos. L}{(1 - e^2 \sin.^2 L)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots (6).$$

All the values which are here obtained suppose x to be taken on the major axis of the ellipse; but if the minor axis were referred to, the method of calculation would be the same for the values of the lines A M, and C M, &c. To prove this, let

$$a^2\,x^2 + b^2\,y^2 = a^2 b^2$$

be the equation of the ellipse, x now being taken on the less axis, C P. If we make the normal A M$=n'$, we shall evidently have for the point A,

$$y^2 = n'^2 \cos.^2 L\,;$$

but, from the equation (2) by changing a for b, and *vice versa*, as well as the sine for the cosine, we shall have

$$A\,M = n' = \frac{a^2}{b}\left[1 - \left(\frac{b^2 - a^2}{b^2}\right) \cos.^2 L\right]^{-\frac{1}{2}};$$

and since we have always

$$a = 1,\ b^2 - 1 = -e^2,\ b = (1 - e^2)^{\frac{1}{2}}$$

it follows that

$$n' = \frac{1}{(1 - e^2)^{\frac{1}{2}}\left[1 + \frac{e^2}{1 - e^2}\,(1 - \sin.^2 L)\right]^{\frac{1}{2}}}$$

$$= \frac{1}{(1 - e^2 \sin.^2 L)^{\frac{1}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

On the other hand $y^2 = n'^2 \cos.^2 L$, therefore

$$y^2 = \frac{\cos.^2 L}{1 - e^2 \sin.^2 L}\,;$$

and as the actual equation of the ellipse gives $y^2 =$

$\frac{(1-e^2)-x^2}{1-e^2}$, by equating these two values, we shall have

$$x^2=\frac{(1-e^2)^2 \sin.^2 L}{1-e^2 \sin.^2 L}.$$

Besides, it has been found above that $C\,L=\frac{a^2-b^2}{a^2}\,x'$; thus, in the present case we have

$$C\,M=\frac{b^2-a^2}{b^2}\,x=\frac{-e^2}{1-e^2}\,x.$$

By substituting for x its value obtained from the preceding equation, we shall have

$$C\,M=\frac{-e^2 \sin.\,L}{(1-e^2 \sin.^2 L)^{\frac{1}{2}}}=-e^2 n' \sin.\,L \ldots\ldots\ldots (8)$$

As to the value of A C, it is evidently represented by $\sqrt{x^2+y^2}$; thus, whether we make use of the above values of x^2 and y^2, or have recourse to the equations (4) and (5), we shall obtain

$$A\,C=r=\left[1-\frac{e^2(1-e^2)\sin.^2 L}{1-e^2 \sin.^2 L}\right]^{\frac{1}{2}} \ldots\ldots\ldots (9).$$

This last formula may be brought to a more simple form, as follows.

If we suppose the ellipsoid circumscribed by a sphere which has for its radius that of the equator, the angle $a\,C\,E=F\,a\,T=\lambda$ will be the latitude of the point a on the sphere; but the points a and A have the same absciss C F; therefore, if we make $A\,F=y'$ and $a\,F=y''$, the equations of the circle and of the ellipse will be respectively

$$y''^2=1-x^2,$$
$$y'^2=b^2(1-x^2);$$

from which, by expunging x^2, we obtain

$$y'^2=b^2 y''^2.$$

But aF is the sine of λ, since $aC=1$; therefore

$$\sin.^2\lambda = \frac{(1-e^2)\sin.^2 L}{1-e^2\sin.^2 L};$$

from this the equation (9) becomes

$$AC=(1-e^2\sin.^2\lambda)^{\frac{1}{2}} \ldots\ldots\ldots (10).$$

We might now find the value of λ; but this Delambre has done in a manner that leaves nothing to be desired (see page 70 of his Memoir already cited).

V. It may be useful to know the nature of the curve formed upon the spheroid by a plane perpendicular to that of the meridian: this research requires first, that we have the equation of the surface of the spheroid of revolution; then by referring that surface to co-ordinates taken in the cutting plane, we shall obtain an equation between two indeterminate quantities only, and we shall thus have the equation of the curve made by the intersection.

To find the equation of a spheroid generated by the revolution of an ellipse about its shorter axis, it must be considered that the generating curve being plane, its equations will be

$$\left.\begin{array}{r} a^2y^2+b^2x^2=a^2b^2 \\ z=o \end{array}\right\} \text{(A)}$$

We shall have besides,

$$\left.\begin{array}{r} y=\alpha \\ x^2+y^2+z^2=f(\alpha) \end{array}\right\} \text{(B)},$$

$y=\alpha$ being the equation of a plane perpendicular to the axis of rotation; and the last, that of a sphere, the centre of which is at the origin of the co-ordinates. (*Feuilles d'Analyse de Monge*, No. 6.)

These four equations ought to take place at the same time, that the proposed surface may be a surface of revolution. Therefore, if we exterminate x, y, and z we shall have $a^2b^2 - a^2\alpha^2+b^2\alpha^2=b^2 f(\alpha)$; and by

substituting for α and f (α) their values (B) we shall have

$$b^2z^2+a^2y^2+b^2x^2=a^2b^2 \text{........} (1).$$

for the equation of the surface of the spheroid of revolution.

Now, if A M be the intersection of the cutting plane with that of the meridian, taken for the plane of the x, y, and that these two planes are perpendicular to each other, we must in the equations relative to the transformation of the axes (see *Le Traité des Surfaces du* 2ᵉ *Ordre par Biot*, *No.* 153) make $\theta=1$, and we shall have for all points of the cutting plane

$$x=a'+x' \cos \phi$$
$$y=b'+x' \sin. \varphi$$
$$z=c'+y'.$$

Substituting these values in the equation (1) we shall find by putting, for the sake of simplification,

$$b'=c'=o,$$

$$b^2 y'^2+(a^2 \sin.^2 \varphi+b^2 \cos.^2\varphi)\, x'^2 +2\,a'\,b^2 \cos.\,\varphi\, x' = (a^2 - a'^2)\, b^2 \text{........} (2).$$

an equation of the ellipse, and that of the required intersection.

This equation would be that of the circle if the co-efficients of x'^2 and y'^2 were equal, that is, if

$$b^2=a^2 \sin.^2\varphi+b^2 \cos.^2\varphi.$$

This condition may be fulfilled by making $\varphi=o$, for in this case, sin. $\varphi=o$, and cos. $\varphi=1$. The curve of intersection is therefore a circle when the cutting plane is parallel to the greater axis of the spheroid of revolution.

If it be required that the intersection A M of the cutting plane should coincide with the normal corresponding to the latitude L, we must make $\varphi=\text{L}$, and

$$a'=\text{CL}=\frac{e^2 \cos. \text{L}}{(1-e^2 \sin. \text{L})\,\frac{1}{2}} \text{ (No. 4. } equa. \text{ 6.)}$$

The equation (2,) in which we may likewise make

$a=1$ and $b^2=1-e^2$ will then become, by means of these values,

$$y'^2+\frac{(1-e^2\cos.^2L)}{1-e^2}x'^2+\frac{2\,e^2\cos.^2L}{(1-e^2\sin.^2L)^{\frac{1}{2}}}x'=1-\frac{e^4\cos.^2L}{1-e^2\sin.^2L}.$$

To have the points in which the curve of intersection cuts the axis of the x', we may make $y'=0$, then

$$x'^2+\frac{2\,e^2\cos.^2L\,(1-e^2)}{(1-e^2\cos.^2L)\,(1-e^2\sin.^2L)^{\frac{1}{2}}}x'-$$

$$\left(1-\frac{e^4\cos.^2L}{1-e^2\sin.^2L}\right)\cdot\frac{(1\,e^2)}{(1-e^2\cos.^2L)}=0$$

If the two roots of this equation be denoted by $x_{,}\ x_{,,}$ we shall have

$$x_{,}=\frac{1-e^2}{(1-e^2\sin.^2L)^{\frac{1}{2}}}$$

$$\text{and } x_{,,}=-\left(\frac{1+e^2\cos.^2L}{1-e^2\cos.^2L}\right)\frac{(1-e^2)}{(1-e^2\sin.^2L)^{\frac{1}{2}}}$$

The first root is the expression for the normal obtained above, and the second root is the value of the opposite normal.

When $L=q$, we have evidently

$$x_{,}=\sqrt{(1-e^2)}=b,\text{ and } x_{,,}=-\sqrt{(1-e^2)}=-b.$$

The actual curve of intersection ought not to be confounded with the line which, between the pole and the equator, would be perpendicular to the meridian; for this last has essentially a double curvature in the spheriod. But these two curves differ much less from each other as the ellipticity of the earth diminishes.

By measuring, in the greatest breadth of France, an arc of the perpendicular to the meridian of the observa-*y of Paris*, with the accuracy which characterises the

last measure of an arc of the meridian comprised between Dunkirk and Barcelona, we should have more certain data relative to the curvature of the earth's surface. This is an operation desired by the learned, and of which Government will, doubtless, shortly order the execution.

Calculation of Terrestrial Refraction.

VI. At 18.6 yards below the top of the signal A (*fig.* 55) the zenith distance was observed to be 90° 13′ from the upper extremity of the signal B; reciprocally, at 16.5 yards below the upper extremity of B, the zenith distance to the point A was 89° 56′; the rectilinear distance between the two signals being 31172.8 yards=K, it is required to determine the value of the refraction and of its co-efficient.

To reduce the zenith distances to the tops of the signals we may make use of the formula (5) No. 1

$$d\Delta = \frac{dH \sin.\Delta R''}{B}.$$

Thus with respect to the signal A we have

$$dH = 18.6,\ \Delta = 90^\circ\ 13;$$

and with regard to the signal B,

$$dH' = 16.5,\ \Delta' = 89^\circ\ 56:$$

Log. dH=1.2695129	Log. $d\ H'$ =1.2174839
Log.sin.Δ=9.9999969	Log. sin. Δ' =9.9999997
Com.log.B=5.5062243	Com.log.B =5.5062243
log.R″=5.3144251	log. R″=5.3144251
Log. $d\Delta$ =2.0901592=123″07	Log.$d\Delta'$ =2.0381330= =109″.18

	$\Delta = 90^\circ\ 13'$		$\Delta' = 89^\circ\ 56'$
	$d\Delta = 0\quad 2\ 3''.07$		$d\Delta' = 0\quad 1\ 49''.18$
Reduction to the Summit A	$90^\circ\ 15'3''.07 = \delta$	Reduction to the Summit B	$89^\circ\ 57'49''.18 = \delta'$

After the apparent zenith distances have been reduced to the summits of the signals, the following formulæ are to be employed in obtaining the refraction and its coefficient.

$$r = \tfrac{1}{2}C - \tfrac{1}{2}(\delta + \delta' - 180),\ n = \frac{r}{C}$$

These formulæ require the angle C at the centre of the earth to be known and expressed in parts of the sexagesimal degree. Now the distance K between the two signals A and B=31172.8 yards; and the logarithm of the equatorial radius ρ in yards=6.8433891. Thus we shall have the angle C by the formula (No. 10) or solely by the assistance of $\frac{KR''}{\rho}$:

$$\begin{array}{ll} \text{Log. K} \ldots\ldots & =4.4937757 \\ \text{Log. R}'' \ldots & =5.3144251 \\ \text{Comp.log. } \rho & =\underline{3.1566109} \\ & 2.9648117 = 922''.17 = C. \end{array}$$

Therefore the distance A B answers to the arc 922''.17 or 15' 22''.17.

The following is the calculation of r.

$$\begin{array}{rl} \delta & = 90^\circ\ 15'\ \ 3''.17 \\ \delta' & = \underline{89\ \ 57\ \ 49\ .18} \\ \delta+\delta'-180^\circ & = \ \ 0^\circ\ 12'\ 52''.25 \\ \dfrac{\delta+\delta'-180^\circ}{2} & = \ \ 0\ \ \ 6\ \ 26\ .125 \\ \tfrac{1}{2}C & = \ \ 0\ \ \ 7\ \ 41\ .085 \\ r & = \ \ 0\ \ \ 1\ \ 14\ .96 \text{ or } 74''.96 \end{array}$$

$$\begin{array}{r} \text{Log.}\,r = 1.8748296 \\ \text{Log. C} = \underline{2.9648110} \\ \text{Log.}\,n = \overline{2}.9100186 \end{array}$$

Therefore $n = .081286 = \frac{1}{12}$ nearly.

Thus, when the refraction is constant, we have

$$r = n\,C = (.08), C.$$

Calculation of the Difference of Level.

VII. All things remaining as in the preceding example, required the difference of level of the tops of the two signals A and B (*fig.* 55). This difference is given by the exact formula, No. 2, by supposing that we know the two apparent zenith distances δ and δ':

$$H=\frac{K\sin.\frac{1}{2}(\delta'-\delta)}{\cos.\frac{1}{2}(\delta'-\delta+C)}$$

δ'........$=$ $89°\ 57'\ 49''$

δ ....$=$ $\underline{90.15\ \ 3}$ Log. K...... $=$ 4.4937757

$\delta'-\delta$ ....$=-0°.17'\ 14''$ Log.sin.$\frac{1}{2}(\delta'-\delta)=-7.3990650$

C ...$=+0.15\ 22$ Comp. log. cos. $\frac{1}{2}$

$(\delta'-\delta+C)=+\underline{0.0000000}$

$\delta'-\delta+C=-0°.\ 1'25''$ $\qquad -\bar{1}.8928407$

Therefore H$=-78.134$ yds.

Let us now find this difference of level by the approximating formula H$=$K tang. $\frac{1}{2}(\delta'-\delta)$;

Log. K...............$=$ 4.4937757

Log. tang. $\left(\frac{\delta'-\delta}{2}\right)=-\underline{7.3990663}$

$-\bar{1}.8928420=-78.135$ yds.

It follows from this that the point B is lower than A by 78.135 yds.

If we know only the zenith distance δ, taken at the point A, it will be necessary to recur to the exact formula (No. 2,) or to this approximating one,

$$H=K\ \cot.(\delta+r-\tfrac{1}{2}C).$$

In this case we have

$$H=K\ \cot.(\delta+.08\,C-.5\,C)=K\ \cot.(\delta-.42\,C).$$

$\delta=90°.15'\ 3''$ Log. .... K$=$ 4.4937757

$-.42\,C=\underline{0.\ \ 6\ 27}-$log.tan.$8'\ 36''=-\underline{7.3982255}$

$\delta-.42\,C=90°.\ 8'\ 36''$ $\qquad -\bar{1}.8920012=$

$=-77.984$ yds.

Finally, if we had only δ', the required difference of level would be obtained from the formula

$$H = -K \cot. (\delta' + r - \tfrac{1}{2} C)$$

$\delta' = 89°.57'49''$ Log. K........ $= 4.4937757$

$-.42\,C = -0.\ 6\ 27$ —log. tang. $8'\,38'' = -7.3999055$

$89°.51'22''$ $\qquad -1.8936812$

$= -78.286$ yds.

And by taking the mean of these two last results we shall obtain that which precedes them; viz.—78.135 yds.

After having thus found the difference of level of two points from which the zenith distances had been reciprocally observed, let us now investigate that of two inaccessible objects, the rectilinear distances of which, from the centre of the station, are known.

From the point A (*fig.* 59), the zenith distances of the points D and C have been observed; one being 90° 19′ 49″ and the other 90° 48′. Required the difference of level of those two points, knowing also that their distances from the station A are respectively 30739.1 and 7052.1 yards, and that the co-efficient of the refraction was at that time 0.08. We have

$K = 30739.1$ yds $\qquad K_{,} = 7052.1$ yds

$\delta = 90°\ 19'\ 49''$ $\qquad \delta_{,} = 90°\ 48'$

$N = K \cot. (\delta + r - \tfrac{1}{2} C)$ $\qquad N_{,} = K_{,} \cot. (\delta_{,} + r_{,} - \tfrac{1}{2} C_{,})$.

Thus the difference of level required will be $N - N_{,}$:

Log. K $= 4.4876916$	Log. $K_{,} = 3.8483179$
Comp. log. $\varrho = 3.1566109$	Comp. log. $\varrho = 3.1566109$
Log. $R'' = 5.3144251$	log. $R'' = 5.3144251$
Log. C $= 2.9587276 = 909''.343$	log. $C_{,} = 2.3193539$
	$= 208''.62$

$\delta = 90°.19'.49''$	$\delta_{,} = 90°\ 48'\ 0''$
$-.42\,C = 0.\ 6\ 22$	$-.42\,C_{,} = 0\ \ 1\ 28$
$90°.13'.27'$	$90°\ 46'32''$

$$\begin{array}{lr} \text{Log. K} = & 4.4876916 \\ -\text{log. tang. } 13' 27'' = & -7.5924506 \\ \hline & -2.0801422 \\ = -120.27 \text{ yds} = \text{N} & \end{array} \qquad \begin{array}{lr} \text{Log. K}_{,} = & 3.8483179 \\ -\text{log. tang. } 46' 32'' = & -8.1315168 \\ \hline & -1.9798347 \\ = -95.463 \text{ yds} = \text{N}_{,} & \end{array}$$

therefore the difference of level of the two points C and D $=\text{N}-\text{N}_{,}=-24.807$ yds, very nearly.

If the same points C and D had been observed at the other station B, we should have had the means of verification; for, by working the same as above, we ought to obtain the same result; or, if the two results differ very little from each other, it will be proper to take the mean, if there be not any reason for preferring one of them to the other. It was by similar reciprocal operations that the last measured base in Bavaria was levelled by Mr. Henry, astronomer of the general Depôt of war.

Calculation of the Height of a Place from which the Horizon of the Sea can be seen.

VIII. From the top of a mountain the distance between the zenith and the horizon of the sea was found 99° 19′ 8″.5, required the height of the centre of observation above the level of the sea.

We have (formula 5, No. 2):

$$\text{H} = \tfrac{1}{2}\,\varrho\,(1+n)^2 \,\text{tang.}^2\,(\delta - 90^\circ)$$

$$\begin{array}{ll} \delta = 90^\circ\ 19'\ 8''.5 & \text{Log. } \tfrac{1}{2} \cdots\cdots = -1.6989700 \\ -90^\circ & \text{Log. } \varrho \cdots\cdots = \ \ 6.8433891 \\ \hline 0^\circ.\ 19'\ 8''.5 & \text{Log. } (1.08)^2 \cdots = \ \ 0.0668475 \\ & \text{Log. tang.}^2\,(\delta-90^\circ) = \ \ 15.4914206 \\ & \qquad\qquad\qquad \overline{2.1006272} = \\ & \qquad\qquad\qquad = 126.075 \text{ yds.} \end{array}$$

Hence the centre of the instrument with which the observation was made, was 126.075 yds above the horizon of the sea.

Demonstrations of the Formulæ proposed by Legendre, for calculating the Latitudes, Longitudes, and Azimuths of Terrestrial Objects.

IX. We are indebted to Dionis-du-Sejour for a very ingenious and exact method of finding the latitude and longitude of a known place on the earth, by its distances from the meridian of another known place, and from its perpendicular. Nevertheless, this method is neither so simple nor so direct as the nature of the problem admits. Those which Legendre and Delambre have lately given on the occasion of the new measure of an arc of the meridian, are free from all objections, and for this reason appear to deserve the preference, wherefore most of the engineers employed in geodesic operations of the first order, make particular use of them.

Such is the motive which induces us to give the demonstrations of some of the formulæ which Legendre has only enunciated in his Memoirs published in 1799. These formulæ, which relate to the present subject, require only the looking out of a very few logarithms, and all have common elements which render the calculations susceptible of being quickly performed.

The following series may be employed, which are too well known to require their demonstrations to be repeated:

$$\left.\begin{aligned}
\sin. x &= x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \dots \\
\cos. x &= 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \dots \\
\text{tang}. x &= x + \frac{x^3}{1.3} + \frac{2x^5}{1.3.5} + \dots
\end{aligned}\right\} \dots (A).$$

There will also be occasion to convert into parts of a quadrant or degrees, an arc given in parts of the radius taken for unity: this operation is founded upon the following consideration:

If we suppose the radius to be applied to the circumference, it will intercept a number of parts of the quadrant which may be obtained by means of this proportion,

$$\pi : 1 :: 2q : \mathrm{R} = \frac{2q}{\pi};$$

π denoting the semi-circumference of a circle whose radius is 1, q a quadrant, and R an arc equal to the radius.

In practice q depends upon the system of the division which is adopted: thus on the hypothesis that $q=90°$, the radius will be expressed in sexagesimal minutes, by

$$\mathrm{R}' = \frac{10800}{\pi} = 3437'.74$$

the logarithm of which is $= 3.5362738$; because $\pi = 3.141592$ and $2\,q = 180° = 10800'$.

In the same manner the radius will be expressed in seconds by

$$\mathrm{R}'' = \frac{648000}{\pi} = 206262''.3,$$

the logarithm of which is 5.3144251.

Thus, any arc whatever given in parts of the radius considered as unity, will be expressed in seconds, for example, by multiplying it by R″; and an arc given in seconds will be converted into parts of the radius, by dividing it by R″.

The calculation of latitudes, longitudes, and azimuths is derived, as will shortly be seen, from the complete solution of a right-angled spherical triangle, one side of which, adjacent to the right angle, is very small with

respect to the other two. It is therefore essential to give this solution as an introduction: for this purpose, let the three angles, as usual, be denoted by A, B, C, and the sides opposite them by a, b, c, respectively.

If A be the right angle, and b the adjacent side which is supposed to be very small with respect to the radius of the sphere upon the surface of which the triangle A B C is situated, we shall have, by the known principles of the exact solution of spherical triangles,

$$\left.\begin{array}{l} \cos. a = \cos. b \cos. c \\ \text{tang. } B = \dfrac{\text{tang. } b}{\sin. c} \\ \text{tang. } C = \dfrac{\text{tang. } c}{\sin. b} \end{array}\right\} \ldots\ldots(B).$$

It follows from the hypothesis laid down above, that the arcs a and c differ very little from each other. Let then $a=c+x$; the substitution of this value in the first of the preceding equations, will give

$$\cos. (c+x)=\cos. b \cos. c;$$

developing the first member it will become

$$\cos. c \cos. x - \sin. c \sin. x = \cos. b \cos. c.$$

But x being very small, we have, by taking only the first terms of the series (A), $\sin. x = x$, $\cos. x = 1$; therefore

$$\cos. c - x \sin. c = \cos. b \cos. c$$

and because b, which is likewise very small, has for its cosine $1-\frac{b^2}{2}+\ldots\ldots$ It follows that

$$\cos. c - x \sin. c = \cos. c\left(1-\frac{b^2}{2}\right).$$

Therefore $x = \dfrac{b^2}{2}\dfrac{\cos. c}{\sin. c} = \frac{1}{2} b^2 \cot c;$

and to have x in seconds

$$x=\tfrac{1}{2}\,\mathrm{R}''\,b^2\cot.\,c\,;$$

therefore lastly

$$a=c+\tfrac{1}{2}\,\mathrm{R}''\,b^2\cot.\,c\text{-------------------------}(1).$$

Such is the value of the hypothenuse in a function of the other two sides; if, on the contrary, c were required in a function of a and b, it might be obtained by a process similar to the preceding. In fact, let $c=a-x'$, we shall have

$$\cos.\,a=\cos.\,b\,\cos.\,(a-x')\,;$$

when developed it will become

$$\cos.\,a=\cos.\,b\,(\cos.\,a+x'\sin.\,a),$$

or else

$$\cot.\,a=\cos.\,b\,(\cot.\,a+x')=\left(1-\frac{b^2}{2}\right)(\cot.\,a+x'),$$

from which we obtain, by neglecting the fourth powers of b,

$$x'=\frac{b^2}{2}\cot.\,a\left(1-\frac{b^2}{2}\right)^{-1}=\tfrac{1}{2}\,b^2\cot.\,a\,;$$

therefore,

$$c=a-\tfrac{1}{2}\,\mathrm{R}''\,b^2\cot.\,a\text{-------------------------}(2.)$$

Now, let us proceed to the developement of the second equation (B) which, by means of the values of the tangents B and b, becomes

$$\mathrm{B}=\frac{b}{\sin.\,c}\left(1+\frac{b^2}{3}\right)-\frac{\mathrm{B}^3}{3}.$$

The angle B and its opposite side being very small, we shall sensibly have

$$\mathrm{B}=\frac{b}{\sin.\,c},\ \text{and}\ \mathrm{B}^3=\frac{b^3}{\sin.^3 c}\,;$$

therefore the preceding equation will take this form,

$$\mathrm{B}=\frac{b}{\sin.\,c}\left(1+\frac{b^2\,(\sin.^2 c-1}{3\sin.^2 c}\right);$$

from which we conclude that

$$B=\frac{R''b}{\sin. c}-\frac{R''b^3\cos.^2 c}{3\sin.^3 c} \text{ .... } (3).$$

There only remains the third equation (B) to be treated. Now, if we make in it $C=q-y$, y being supposed very small, it will become

$$\text{tang. }(q-y)=\cot. y=\frac{\text{tang. } c}{\sin. b};$$

from which

$$\text{tang. } y=\cot. c \sin. b.$$

Substituting for the tang. y and sin. b, their values deduced from the series (A) we shall have

$$y+\frac{y^3}{3}=b\cot. c-\frac{b^3}{6}\cot. c.$$

This equation shews that we have very nearly $y=b$ $\cot. c$, or $\frac{y^3}{3}=\frac{b^3}{3}\cot.^3 c$;

Thus the same equation will take the following form,

$$y=b\cot. c-\frac{b^3}{3}\cot. c\,(\tfrac{1}{2}+\cot.^2 c):$$

we have therefore

$$C=q-R''b\cot. c+R''\frac{b^3}{3}\cot. c\,(\tfrac{1}{2}+\cot.^2 c) \text{ ..... } (4).$$

This is the value of the angle C given in a function of the two sides about the right angle. The relation which subsists between C, a, b, may now be found.

First, from the preceding solution, we have $c=a-\frac{1}{2}$ $b^2\cot. a$; thus

$$C=q-b\cot.(a-\tfrac{1}{2}b^2\cot. a)+$$

$$\frac{b^3}{3}\cot.(a-\tfrac{1}{2}b^2\cot. a)\left[\tfrac{1}{2}+\cot.^2(a-\frac{b^2}{2}\cot. a)\right];$$

Developing this on the hypothesis that the second term of the value of c is very small; and neglecting the

fourth and higher powers of b, we shall have, because the cot. a tang. $a=1$ and that $\frac{1}{\text{tang.}} = \text{cot.}$

$$C = q - \left(b + \frac{b^3}{2}\right)\left(\text{cot. } a + \frac{b^2}{2}\text{cot.}^3 a\right)$$
$$+\tfrac{1}{3} b^3 \text{ cot. } a\,(\tfrac{1}{2}+\text{cot.}^2 a),$$

and then

$$C=q-R'' b \text{ cot. } a - \tfrac{1}{3} R'' b^2 \text{ cot. } a\,(1+\tfrac{1}{2}\text{ cot.}^2 a)..(5).$$

On the Meridian, and the Perpendicular to that Meridian.

X. When all the triangles which form the outlines of a great geographical map have been calculated according to the principles already explained, the vertices of their angles are to be referred to two lines perpendicular to each other: the one exactly represents the meridian of the principal place of the map, and is called the *meridian;* the other passes through the same place, and is called the *perpendicular*.

To these two lines, considered as axes to which the abscisses and ordinates are drawn, are referred, like the points of a curve, all those of the country which the triangles embrace. If the map is only of small extent, the distances of the objects from the perpendicular, converted into parts of the quadrant, will express the latitudes with sufficient accuracy; and the distances from the meridian, converted into the same parts, will express the longitudes relative to these two axes. Then it will be easy to know the absolute latitudes and longitudes of the same objects, that is their situations on the globe. But this method is not admissible when the points referred to the meridian and its perpendicular are very distant from them, and it is essential, in this case, to take into

the account the ellipticity of the earth; such is the subject of the following formulæ.

Let P be the pole of the earth, (*fig.* 60), P A and P B two elliptic meridians; let also the known latitude of the point A be denoted by L: it is required to find the latitude of the point B, situated on the arc A B, perpendicular to P A; its longitude, and the angle P B A or the azimuth of A, observed from B.

Suppose at the two points A and B the vertical lines A M and B N, and make $AB=y$, $AM=r$, $NB=r'$. The small arc y having r for its radius of curvature, it follows that a similar arc φ, the radius of which is 1, will be $\frac{y}{r}$.

This being premised, if, from the point M as a centre, and with $Mb=1$, as a radius, the arcs ab, ap, pb, be described, we shall have the spherical triangle pab, in which the side $pa=q-L$, the side $ab=\varphi$, and the contained angle $pab=q$, are known by the hypothesis; now, according to the principles of trigonometry,

$$\cos.\,(pb)=\sin.\,L\,\cos.\,\varphi,$$

$$\text{tang. } P=\frac{\text{tang. }\varphi}{\cos.\,L},$$

$$\text{tang. } b=\frac{\cot.\,L}{\sin.\,\varphi}.$$

These formulæ are the same as those which have been denoted above by (B), No. 9; by comparing them together it will easily be seen that

$a=(pb)$, $b=\varphi$, $c=q-L$, $B=P$, $C=b$.

It follows from this, and from the developements performed in the number above cited, that

$$(pb)=q-L+\tfrac{1}{2}\varphi^2\,\text{tang.}\,L,$$

$$P = \frac{\varphi}{\cos. L} - \tfrac{1}{3}\varphi^3 \frac{\sin.^2 L}{\cos.^3 L},$$

$$b = q \quad \varphi \text{ tang. } L + \tfrac{1}{3}\varphi \text{ tang. } L\left(\tfrac{1}{2} + \text{tang.}^2 L\right).$$

From the value of $(p\,b)$ we deduce, for an approximation to the latitude of B,

$$q - (p\,b) = L - \tfrac{1}{2}\varphi^2 \text{ tang. } L.$$

The value of P is the difference of longitude between the two points A and B, and the value of b is the required azimuth P B A.

To obtain the latitude of the point B more exactly, it may be remarked that it is equal to the complement of the angle P N B or of the angle P M B + N B M $= p\,b +$ N B M; but because the angle N A M is very nearly equal to the angle N B M, we shall have

$$N B M = \frac{M N \cos. L}{r} = \psi.$$

M N is easily obtained by means of the formula (8) of No. 4; for if we make $n' = r$, we shall have, independent of the sign, C M $= e^2 r \sin. L$. In like manner for the point B, the latitude of which is L′, we have C N $= e^2 r' \sin. L'$; thus, very nearly,

$$M N = e^2 r\,(\sin. L - \sin. L').$$

Likewise, from the trigonometrical formulæ, we have

$$\sin. L - \sin. L' = 2 \sin.\left(\frac{L - L'}{2}\right) \cos.\left(\frac{L + L'}{2}\right);$$

from which it follows, that

$$M N = 2\, e^2 r \sin. \frac{L - L'}{2} \cos. \frac{L + L'}{2}.$$

On the supposition that $L - L'$ is very small, the arc may be taken for the sine, and the cos. L substituted instead of $\cos. \frac{L + L'}{2}$. It then becomes

$$M N = e^2 r\,(L - L') \cos. L.$$

Now, if it be considered that the approximative latitude of B is $L' = L - \frac{1}{2}\varphi^2$ tang. L, we shall have

$$M N = e^2 r (L - L + \tfrac{1}{2}\varphi^2 \text{ tang. } L) \text{ cos. } L.$$
$$= \tfrac{1}{2} e^2 r \varphi^2 \text{ tang. } L \text{ cos. } L = \tfrac{1}{2} e^2 r \varphi^2 \text{ sin. } L.$$

Thence it follows that the angle $N B M = \frac{1}{2} e^2 \varphi^2$ sin. L cos. L, and consequently that the true latitude of

$$B = q - (p\,b) - \psi \text{ will be}$$
$$L' = L - \tfrac{1}{2}\varphi^2 \text{ tang. } L - \tfrac{1}{2} e^2 \varphi^2 \text{ sin. } L \text{ cos. } L.$$

Strictly speaking, the azimuth above calculated is only an approximation ; for the true azimuth of A B, with respect to the meridian P B, is the angle formed by the two planes P N B and A B N, since their common section or the vertical from the point B is the right line B N : but we shall demonstrate, agreeably to Delambre, that the correction of the azimuth is insensible.

If the point B be considered as the centre of a sphere, the three planes A B M, N B M, A B N, (*fig.* 60), form by their intersection with the surface, a spherical triangle a' M n ; now in this triangle we know the angle a' M n ; which is the calculated azimuth. We also know the arc n M or the angle $N B M = \psi$, which is the correction of the latitude ; and because the angle $A M B = \varphi$ we have, by considering the triangle A B M as isosceles, the arc $a' M = q - \frac{\varphi}{2}$; therefore, if ξ denote the correction of the azimuth, and Z' the true azimuth reckoned from the north, in which case $Z' - \xi = a' M n$ will be the approximative azimuth, we shall, as is well known, have

$$-\text{tang.} Z' = \frac{\text{sin. } (Z' - \xi)}{\text{tang. } \frac{\varphi}{2} \text{ sin. } \psi - \text{cos. } (Z' - \xi) \text{ cos. } \psi}$$

from which

$$\text{tang.}\,(Z'-\xi)=\text{tang.}\,Z'\cos.\psi-\frac{\text{tang.}\,Z'\,\text{tang.}\frac{\phi}{2}\sin.\psi}{\cos.(Z'-\xi)};$$

adding the tangent of Z' to both sides, and changing all the signs, it will become

$$\text{tang.}\,Z'-\text{tang.}\,(Z'-\xi)=\text{tang.}\,Z'(1-\cos.\psi)+\frac{\text{tang.}\,Z'\,\text{tang.}\frac{\phi}{2}\sin.\psi}{\cos.(Z'-\xi)}$$

If we clear the denominator, the first member may be put under this form,

$\cos.\xi\,(\text{tang.}\,Z'\cos.Z'-\sin.Z')+\sin.\xi\,(\text{tang.}\,Z'\sin.Z'+\cos.Z')$,

and be reduced to $\frac{\sin.\xi}{\cos.Z'}$; because $\text{tang.}\,Z'=\frac{\sin.Z'}{\cos.Z'}$ and $\sin.^2 Z'+\cos.^2 Z'=1$. Therefore, by observing that $1-\cos.\psi=2\sin.^2\frac{1}{2}\psi$, and that ξ is very small, the preceding equation becomes,

$\xi=\sin.Z'\,\text{tang.}\,\frac{1}{2}\phi\sin.\psi+2\sin.Z'\sin.^2\frac{1}{2}\psi\cos.(Z'-\xi)$.

It appears from this formula that ξ is less than ψ, which is itself very small: the correction of the azimuth may therefore always be neglected, which might otherwise be calculated, since the true azimuth = the approximative azimuth $+\xi$.

It results from what precedes, that, if L be the latitude of the point A, (*fig.* 60), L′ the latitude of the point B, y the perpendicular distance AB from the meridian of A, and r the radius of the earth, or the normal at the point A, we shall have, by denoting the number of sexagesimal seconds in the radius by R,

$$L'=L-\tfrac{1}{2}R\frac{y^2}{r^2}\,\text{tang.}\,L-\tfrac{1}{2}R\,e^2\frac{y^2}{r^2}\sin.L\cos.L\;\ldots.(a).$$

and reciprocally

$$L = L' + \tfrac{1}{2} R \frac{y^2}{r^2} \text{tang. } L' + \tfrac{1}{2} R e^2 \frac{y^2}{r^2} \text{sin. } L' \text{cos. } L' \ldots (b).$$

As e denotes the ellipticity of the earth, it is evident that the last term of these formulæ may almost always be neglected.

The same being premised, we shall have the difference of longitude of the points A and B by this formula.

$$P = \frac{R\,y}{r\,\text{cos. } L} \left(1 - \tfrac{1}{3} \frac{y^2}{r^2} \text{tang.}^2 L\right) \ldots\ldots\ldots (c).$$

and the azimuth of the arc B A or angle P B A will be

$$Z' = q - \frac{R\,y}{r} \text{tang. } L + \tfrac{1}{3} \frac{R y^3}{r^3} \text{tang.} L (\tfrac{1}{2} + \text{tang.}^2 L) (d).$$

If L′ only be known, the formula 5 (No. 9) must be used, and we shall have

$$Z' = q - R \frac{y}{r} \text{ tang. } L' - \tfrac{1}{3} R \frac{y^3}{r^3} \text{ tang. } L' (1 + \tfrac{1}{2} \text{tang.}^2 L') \ldots\ldots\ldots\ldots (e).$$

It often happens that we do not immediately know the latitude L of the point A; but then we necessarily know the distance x from this point to the perpendicular of the principal place on the map: in this case, x must be reduced into parts of a degree, and, according as it is north or south of this perpendicular, its value must be added or subtracted from the latitude of the principal place, in order to obtain that of the point A. This reduction may be deduced with sufficient accuracy from the value of φ expressed in a function of the latitude L of the point A; for, because

$\varphi = \frac{y}{r}$ and $r = \frac{1}{(1 - e^2 \text{sin.}^2 L)^{\frac{1}{2}}}$ (formula 7, *No.* 4) we have

$$\varphi = y\,(1 - e^2 \text{sin.}^2 L)^{\frac{1}{2}} = y\,(1 - \tfrac{1}{2} e^2 \text{sin.}^2 L \ldots).$$

In this equation, y is understood as making part of the radius of the equator $= 1$; but in practice, this ra-

dius$=\rho$; hence it will be necessary to substitute $\frac{y}{\rho}$ for y, which will be reduced into seconds by this formula

$$\varphi = \frac{R''y}{\rho}(1 - \tfrac{1}{2}e^2 \sin.^2 L);$$

and by substituting x for y we shall in the same manner have the arc x reduced into parts of a degree.

For the latitude $L = \frac{1}{4}\pi = \frac{1}{2}q$, we have $\sin.^2 L = \frac{1}{2}$, and then

$$\varphi = \frac{R''y}{\rho}(1 - \tfrac{1}{4}e^2).$$

Lastly, on the hypothesis that the earth is spherical, we shall have $e = 0$, therefore

$$\varphi = \frac{R''y}{\rho}.$$

Application of the preceding Formulæ.

XI. Let $L = 39° 41' 24''$, the latitude of the point A (*fig.* 60) and $y = 25960$ yards, the perpendicular A B to the meridian A P; required the latitude of the point B, the difference of longitude of A and B, and the azimuth of A observed at the point B, or the angle A B P.

The radius of curvature r of the arc A B may be calculated by the formula

$$r = \frac{\rho}{(1 - e^2 \sin.^2 L)^{\frac{1}{2}}} \text{ (No. 4) thus}$$

$\text{Log.}\, e^2 = \log.\, 0.00597906 = -\,3.7766330$

$\text{Log.}\sin.^2 L = \log.\sin^2 39°41'24'' = \underline{19.6105034}$ $\quad 1.0000000$

$-\,3.3871364 = \underline{0.0024386}$

$1 - e^2 \sin^2 L = 0.9975614$

$\text{Log.}\, \rho$ .... $= 6.8433891$

$\text{Com. log.}\, (1 - e^2 \sin.^2 L)^{\frac{1}{2}} = \underline{0.0005302}$

$\text{Log.}\, r = 6.8439193$

Having found the logarithm of the radius, we shall now work out the latitude of B by means of the formula (a).

$$\begin{array}{rl}
\text{Log.}\,y = & 4.4143047 \\
\text{Comp. log.}\,r = & 3.1560807 \\
 & \overline{7.5703854} \\
\text{Log.}\left(\dfrac{y^2}{r^2}\right) = & -5.1407708 \\
\text{Log. R}'' \ldots = & 5.3144251 \\
\text{Log. } 0.5 = & -1.6989700 \\
 & \overline{0.1541659} = \text{M.}
\end{array}$$

1st Term of the correction.

$$\begin{array}{rl}
\text{M} = & 0.1541659 \\
\text{Log.tang. L} = & 9.9190369 \\
 & -\overline{0.0732028} = 1''.184
\end{array}$$

2nd Term.

$$\begin{array}{rl}
\text{M} = & 0.1541659 \\
\text{Log.}\,e^2 = & -3.7766330 \\
\text{Log.sin. L} = & 9.8052517 \\
\text{Log.cos. L} = & 9.8862148 \\
 & -\overline{3.6222654} \\
 & = 0''.0042
\end{array}$$

$$\begin{array}{rl}
\text{Latitude L} = & 39^\circ\ 41'\ 24'' \\
\left.\begin{array}{l} \textit{1st}\ \text{Term } 1.184 \\ \textit{2nd}\ \text{Term } 0.0042 \end{array}\right\} = & 0\quad 0\quad 1''.1882 \\
 & \overline{39^\circ\ 41'\ 22''.8118} = \text{latitude L}'.
\end{array}$$

This solution shews that we might have neglected the second term of the correction without incurring any practical inconvenience.

For calculating the difference of longitude, the equation (c) may be used, and we shall have

$$\begin{array}{rl}
\text{Log. } \tfrac{1}{3} \ldots\ldots\ldots = & -1.5228787 \\
\text{Log.}\left(\dfrac{y^2}{r^2}\right) \ldots\ldots = & -5.1407708 \\
\text{Log. tang.}^2\ \text{L} = & \underline{19.8380738} \qquad 1\ldots\ldots\ldots\ldots \\
 & -\overline{6.5017233} = 0.0000031748 \\
 & \qquad\qquad\quad \overline{0.9999968252} = \text{N}
\end{array}$$

Log. N............$=-1.9999984$
Log. R″ ....$= 5.3144251$
Log. $\frac{y}{r}$............$=-3.5703854$
Comp. log. cos. L.....$= 0.1137868$
Log. P ....$= \overline{2.9983941} = 996''.77$

Hence the required difference of longitude is 0° 16′ 37″ nearly.

There still remains to be found the azimuth of A or the angle P B A, which may be done by means of the equation (d), from which results the following computation:

1st Term of correction.

Log. R″....$=5.3144251$ 0.5
Log. $\frac{y}{r}$ $=-3.5703854$ Log. tang.^{2} L $= 19.8380738 = 0.68877$
Log. tang. L $=9.9190369$ Q $=1.18877$
$\overline{2.8038474} = 636''.57$. Subtractive

2nd Term.

Log. $\frac{1}{3}$....$=-1.5228787$
Log. R″..$= 5.3144251$
Log. $\frac{y^3}{r^3}$ ....$=-8.7111562$
Log. Q ...$= 0.0750979$
$-\overline{3.6235579} = 0''.004203$ Additive

We have therefore

1st Term $-636''.57$ } 90°
2nd .... $+ 0''.0042$ } 0° 10′ 36″.5
$= 89\ 49\ 23\ .5$ for the azim. req^d.

The total correction of the azimuth $= 10'\ 36''.5$ very nearly is also called the angle of convergence of the meridians P A, P B.

Developement of Legendre's Method for Finding the Equations between the Lengths of Terrestrial arcs and the Latitudes of their Extremities.

XII. The determination of the exact length of a meridian has in all times occupied the most distinguished geometers; and when it was proposed to establish the metric system in France, it became again the object of the most learned researches. The nature of this work does not permit us to enter much into details on that important subject; but we shall endeavour to represent clearly one of the most elegant and accurate methods that has ever been proposed relative to the measure of the earth. If a concise explanation would be sufficient for persons who are not much versed in analysis, it would only be necessary to refer them to the excellent memoir of Legendre; but, for such persons, illustrations may not be superfluous; it is therefore thought, that, by re-establishing in a simple manner the intermediate steps of the *calculus* of that distinguished geometer, young mathematicians will more easily comprehend his method.

Let a be the radius C E of the equator (*fig.* 60) b the semi-axis or polar radius C P, v the radius vector A C, and φ the angle P C A.

If C be the origin of the co-ordinates, and C E the axis of the abscisses, we shall evidently have

$$x = v \sin. \varphi, \; y = v \cos. \varphi;$$

and, supposing first that P A E is an elliptic arc, the equation of the points will be

$$a^2 y^2 + b^2 x^2 = a^2 b^2;$$

which being transformed into that of the polar co-ordinates, becomes

$$a^2 v^2 \cos.^2 \varphi + b^2 v^2 \sin.^2 \varphi = a^2 b^2;$$

from which we obtain

$$v^2 = \frac{a^2 b^2}{a^2 \cos.^2\varphi + b^2 \sin.^2\varphi}.$$

Let α be the ellipticity of the earth, that is, the excess of the greater over the less axis, denoting the greater by unity. We shall have on this hypothesis,

$$b = a(1-\alpha);\ \text{from this}$$

$$v^2 = \frac{b^2}{1-(2\alpha-\alpha^2)\sin.\varphi^2};$$

and by taking the square root,

$$v = b\,[1+(\alpha-\tfrac{1}{2}\alpha^2)\sin.^2\varphi + \tfrac{3}{2}\alpha^2\sin.^4\varphi].$$

by limiting ourselves to quantities of the order α^2.

For the sake of abridgment, this equation may be written thus:

$$v = b\,(1 + m\sin.^2\varphi + n\sin.^4\varphi) \text{-----------------} (A).$$

This equation will therefore be identical with the preceding one, if we have

$$m = \alpha - \tfrac{1}{2}\alpha^2,\ n = \tfrac{3}{2}\alpha^2$$

or as m differs very little from α, if $n = \frac{3}{2}m^2$.

Adding together the two equations above, gives

$$m + n = \alpha + \alpha^2;*$$

So that, when we know $m+n$, we shall have the value of the ellipticity.

The most general hypothesis that can be assumed for the nature of the curve of the meridian is expressed by (A). It is confirmed by numerous investigations on the

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* This first result is not absolutely the same as Legendre's, because this geometer has supposed $a = b\,(1+\alpha')$, that is, he has taken the less axis as unity. On this latter hypothesis we should have $m = \alpha' - \frac{3}{2}\alpha'^2$, $n = \frac{3}{2}\alpha'^2$, and $m+n = \alpha'$. At all events, it is evident that if $\alpha' = \frac{1}{\epsilon}$ be the ellipticity relative to the smaller axis, $\frac{1}{\epsilon+1} = \alpha$ will be the ellipticity when referred to the greater.

theory of the equilibrium of fluids; and if the conditionary equation $n=\frac{3}{2}m^2$ is not fulfilled in that circumstance, it will serve at least to enable us to judge how far the figure of the meridian approaches that of an ellipse, and to indicate that n is of the order m^2.

This premised, let S be the arc of the meridian comprised between the two radii vector a, v: and we shall have in general

$$d\,S=\sqrt{(d\,x^2+d\,y^2)}.$$

To render this differential of the arc a function of v and φ, the differential calculus may be applied to the equations $x=v\ \sin.\varphi$, $y=v\cos.\ \varphi$, regarding v and φ as variable quantities, and we shall have

$$d\,x=v\,d\,\varphi\ \cos.\varphi\ +\ d\,v\sin.\ \varphi,$$
$$d\,y=-v\,d\,\varphi\ \sin.\ \varphi+d\,v\ \cos.\ \varphi\ ;$$

which being squared, added together, and reduced, will give

$$d\,x^2+d\,y^2=v^2\,d\,\varphi^2+d\,v^2.$$

It is obvious that the arc S diminishes when the radius vector v increases, thus

$$d\,S=-\sqrt{(v^2\,d\,\varphi^2+d\,v^2)}.$$

Extracting the root, and observing that as $d\,v$ is of the order α, we may stop at the quantities of the order α^2 inclusively, we shall find

$$d\,S=-v\,d\,\varphi-\frac{d\,v^2}{2\,v\ d\,\varphi};$$

but the differential of the equation (A) is

$$d\,v=b\,(2\,m\ \sin.\ \varphi\cos.\ \varphi\ d\,\varphi+4\,n\ \sin.^3\varphi\cos.\ \varphi\ d\,\varphi)\ ;$$

squaring the whole, and dividing by $2\,v\,d\,\varphi$, we shall have

$$\frac{d\,v^2}{2\,v\,d\,\varphi}=\frac{4\,b^2\,m^2\sin.^2\varphi\cos.^2\varphi\,d\,\varphi^2}{2\,b\,d\,\varphi\,(1+m\sin.^2\varphi)}=2b\,d\,\varphi\,m^2\ \sin.^2\varphi\cos.^2\varphi$$

by neglecting the other terms; therefore

$$dS=-v\,d\varphi-\frac{dv^2}{2v\,d\varphi}$$

$$-b\,d\varphi\,(1+m\sin.^2\varphi+n\sin.^4\varphi+2\,m^2\sin.^2\varphi\cos.^2\varphi).$$

In order to integrate this equation, the quantity within the parenthesis may be put under the following form; that is, all the sines may be changed into cosines,

$$1+m\sin.^2\varphi+n\sin.^4\varphi+2\,m^2\sin.^2\varphi\cos.^2\varphi$$

$$=1+m+n+(2\,m^2-2\,n-m)\cos.^2\varphi+(n-2\,m^2)\cos.^4\varphi;$$

or; for the purpose of abridging,

$$=M+N\cos.^2\varphi+P\cos.^4\varphi.$$

If we indicate only the integration of the above formula, we shall have

$$S=-b\,(M\textstyle\int d\varphi+N\int d\varphi\cos.^2\varphi+P\int d\varphi\cos.^4\varphi).$$

But $\int d\varphi=\varphi$

$$\textstyle\int d\varphi\cos.^2\varphi=\frac{1}{2}\int d\varphi\cos.2\varphi+\frac{1}{2}\int d\varphi=\frac{1}{4}\sin.2\varphi+\frac{1}{2}\varphi,$$

$$\textstyle\int d\varphi\cos.^4\varphi=\frac{1}{8}\int d\varphi\cos.4\varphi+\frac{1}{2}\int d\varphi\cos.2\varphi+\frac{3}{8}\int d\varphi$$

$$=\tfrac{1}{32}\sin.4\varphi+\tfrac{1}{4}\sin.2\varphi+\tfrac{3}{8}\varphi;$$

therefore $S=-b\varphi(M+\frac{1}{2}N+\frac{3}{8}P)-b\,(\frac{1}{4}N+\frac{1}{4}P)\sin.2\varphi$

$$-b\,(\tfrac{1}{32}P)\sin.4\varphi+\text{constant quantity}.$$

Restoring the values of M, N, P, we shall have,

finally, $S=-b\,\varphi\left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)$

$$+\,b\,(\tfrac{1}{4}m+\tfrac{1}{4}n)\sin.2\varphi$$

$$+\,b\left(\frac{m^2}{16}-\frac{n}{32}\right)\sin.4\varphi+\text{const. quan.}$$

The constant quantitity is determined by making $\varphi=q$, q denoting a quadrant; now, in this case, $S=0$, $\sin.2\varphi=0$, $\sin.4\varphi=0$; and therefore the constant quantity

$$=q\,b\left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)$$

It follows from this, that

$$\left.\begin{aligned}S=b\left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)(q-\varphi)\\ +\,b\,(\tfrac{1}{4}m+\tfrac{1}{4}n)\sin.2\varphi\\ +\,b\left(\frac{m^2}{16}-\frac{n}{32}\right)\sin.4\varphi\end{aligned}\right\}\cdots\cdots\cdots(B).$$

It is now proper to introduce for the angle φ, the latitude L of the extremity of the arc. To accomplish this, suppose the radius vector C R (*fig.* 61) to be infinitely near to C A; and from the point C as a centre, with a radius A C, describe the arc A Q, which may be considered as sensibly equal to a chord perpendicular to the radius vector A C. Suppose also the right line A R to be a tangent to the curve at the point A, and produce the right lines C A and L A to Z; then it is evident that the angle C A L = Z A V, will be very nearly equal to the angle R A Q.

This being premised, let C A L $=\psi$, we shall have, by preserving the same denominations as above

$$\mathrm{L}=q-\varphi+\psi,\ \text{from which}\ \psi=\varphi+\mathrm{L}-q;$$

therefore, $\text{tang.}\ \psi=\text{tang.}\ (\varphi+\mathrm{L}-q)$.

On the other side, the triangle C A Q, supposed right-angled at A, gives A Q $=v\,d\varphi$; and as the triangle A Q R is sensibly right-angled at Q, and Q R $=dv$, we shall have

$$\text{tang. Q A R or tang.}\ \psi=\frac{d v}{v\, d\varphi}=\text{tang.}\ (\varphi+\mathrm{L}-q).$$

The angle ψ being very small, we may without sensible error make

$$\varphi+\mathrm{L}-q=\frac{d v}{v\, d\varphi}.$$

From this results

$$\mathrm{L}=q-\varphi+\frac{d v}{v d\varphi}=q-\varphi+2\sin.\varphi\cos.\varphi$$
$$[m+(2n-m^2)\sin.^2\varphi],$$

or because

$$2\sin.\varphi\cos.\varphi=\sin. 2\varphi,\ \text{and}\ \sin.^2\varphi=\frac{1-\cos. 2\varphi}{2}$$

$$\mathrm{L}=q-\varphi+(m+n-\tfrac{1}{2}m^2)\sin. 2\varphi$$
$$+\left(\frac{m^2}{4}-\frac{n}{2}\right)\sin. 4\varphi \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{C}).$$

Now, to find φ in a function of L, it may be remarked that we have first, very nearly, $\varphi = q - \mathrm{L}$, consequently

$$\sin. 2\varphi = \sin. (2q - 2\,\mathrm{L}) = \sin. 2\,\mathrm{L}\,;$$
$$\sin. 4\,\varphi = \sin. (4q - 4\,\mathrm{L}) = -\sin. 4\,\mathrm{L}.$$

These two last values being introduced into the equation (C), we shall have, in a manner more approximative,

$$\varphi = q - \mathrm{L} + \left(m + n - \frac{m^2}{2}\right) \sin. 2\,\mathrm{L} - \left(\tfrac{1}{4}\,m^2 - \tfrac{1}{2}n\right) \sin. 4\mathrm{L},$$

or, for abridging, $\varphi = q - \mathrm{L} + x$; x being evidently a very small quantity.

Substituting again this last quantity in the same equation (C), we shall have, for a nearer approximation,

$$\varphi = q - \mathrm{L} + \left(m + n - \frac{m^2}{2}\right) \sin. (2q - 2\,\mathrm{L} + 2\,x)$$
$$+ \left(\tfrac{1}{4}\,m^2 - \tfrac{1}{2}\,n\right) \sin. (4\,q - 4\,\mathrm{L} + 4\,x)\,;$$

But

$$\left.\begin{array}{l} \sin. (2\,q - 2\,\mathrm{L} + 2\,x) = \sin. 2\,\mathrm{L} - 2\,x \cos. 2\,\mathrm{L} \\ \sin. (4\,q - 4\,\mathrm{L} + 4\,x) = -\sin. 4\,\mathrm{L} + 4\,x \cos. 4\,\mathrm{L} \end{array}\right\} \ldots (\mathrm{D})\,;$$

Therefore

$$\varphi = q - \mathrm{L} + \left(m + n - \frac{m^2}{2}\right) (\sin. 2\,\mathrm{L} - 2\,x \cos. 2\,\mathrm{L})$$
$$+ \left(\tfrac{1}{4}\,m^2 - \frac{n}{2}\right) (-\sin. 4\,\mathrm{L} + 4\,x \cos. 4\,\mathrm{L}).$$

Developing, always stopping at quantities of the same order, and eliminating x, we shall obtain, definitively,

$$\varphi = q - \mathrm{L} + \left(m + n - \frac{m^2}{2}\right) \sin. 2\mathrm{L} - \left(\tfrac{5}{4}\,m^2 - \frac{n}{2}\right) \sin. 4\,\mathrm{L},$$

or, for abridging the work, let $\varphi = q - \mathrm{L} + y$.

This value being substituted in the equation denoted by (B,) we shall have, because of the formulæ (D) and from

$$y = \left(m + n - \frac{m^2}{2}\right) \sin. 2\,\mathrm{L} - \left(\tfrac{5}{4}\,m^2 - \frac{n}{2}\right) \sin.\ 4\ \mathrm{L},$$

$$S = b\left[\left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)L - \tfrac{3}{4}\,(m+n)\ \text{sin.}\ 2\,L\right.$$

$$\left.+\ \tfrac{15}{16}\,(m^2-\tfrac{1}{2}\,n)\ \text{sin.}\ 4\,L\right] \ldots\ldots\ldots\ldots\ldots\ldots (E).$$

Let M denote a quadrant of the meridian, we shall have, by making $L = \frac{1}{2}\,\pi$, π being the semi-circumference of a circle whose radius $= 1$,

$$S = M = b\left(1+\frac{m^2}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)\tfrac{1}{2}\,\pi\,;$$

and therefore

$$\frac{M}{\frac{1}{2}b\pi} = \left(1+\frac{m}{2}+\frac{3n}{8}+\frac{m^2}{4}\right) \ldots\ldots\ldots\ldots\ldots\ldots (F)$$

If $m+n$, the co-efficient of sin. 2 L, and $m^2-\dfrac{n}{2}$, the co-efficient of sin. 4 L, be divided by the second member of this equation, and the very small terms neglected, the quotients will be respectively

$$(m+n)\left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)^{-1} = m+n-\frac{m^2}{2},$$

$$\left(m^2-\frac{n}{2}\right)\ \left(1+\frac{m}{2}+\frac{3\,n}{8}+\frac{m^2}{4}\right)^{-1} = m^2-\tfrac{1}{2}\,n\,;$$

thus, very nearly,

$$m+n = \left(m+n-\frac{m^2}{2}\right)\frac{M}{\frac{1}{2}\pi b},$$

$$m^2-\frac{n}{2} = \left(m^2-\frac{n}{2}\right)\frac{M}{\frac{1}{2}\pi b}\ ;$$

from this, the equation (E) will become

$$S = M\left[\frac{L}{\frac{1}{2}\,\pi} - \tfrac{3}{4}\left(\frac{m+n-\frac{1}{2}\,m^2}{\frac{1}{2}\,\pi}\right)\text{sin.}\ 2\,L\right.$$

$$\left.+\ \tfrac{15}{16}\left(\frac{m^2-\frac{n}{2}}{\frac{1}{2}\,\pi}\right)\ \text{sin.}\ 4\ L\ \right];$$

Therefore, if S′ be another arc terminating one way at the latitude L′, and the other at the equator, we shall evidently have

$$S'-S=M\left[\frac{L'-L}{\frac{1}{2}\pi}-\tfrac{3}{4}\left(\frac{m+n-\frac{1}{2}m^2}{\frac{1}{2}\pi}\right)(\sin.\ 2\ L'-\sin.\ 2\ L)+\tfrac{15}{16}\left(\frac{m^2-\frac{1}{2}n}{\frac{1}{2}\pi}\right)(\sin.\ 4\ L'-\sin.\ 4\ L)\right].$$

Such is the equation which gives the relation of any arc $S'-S$ of the meridian, and the latitudes of its extremities. It appears that, from this equation, the length of a quarter of the meridian M, may be immediately obtained when the co-efficients m and n are known.

If, in order to abridge the expression, we make

$$p=\tfrac{3}{4}\left(\frac{m+n-\frac{1}{2}m^2}{\frac{1}{2}\pi}\right),\ q=\tfrac{15}{16}\left(\frac{m^2-\frac{1}{2}n}{\frac{1}{2}\pi}\right),$$

we shall have

$$S'-S=M\left(\frac{L'-L}{\frac{1}{2}\pi}-p\,(\sin.\ 2\ L'-\sin.\ 2\ L)+q\,(\sin.\ 4\ L'-\sin.\ 4\ L)\right)\ldots\ldots\ldots\ldots\ldots(G).$$

In this equation, p and q may be regarded as the unknown co-efficients; and as q is much less than p, there is not any impropriety in neglecting, at first, the term that contains q.

Let us call L, L', L'', L''', L'''', the respective latitudes of the parts P, P', P'', P''', P'''', situated on the same meridian, and suppose $L < L' < L'' < L''' < L''''$; also call S, S', S'', S''', S'''', the arcs of the meridian comprised between the equator and each of these different points. The preceding equation, applied successively to the two arcs, $S\,S''$, $S''\,S''''$ will give, by neglecting q, these two equations,

$$\left.\begin{aligned}S''-S&=M\frac{L''-L}{\frac{1}{2}\pi}-M\,p\,(\sin.\ 2\ L''-\sin.\ 2\ L),\\ S''''-S''&=M\frac{L''''-L''}{\frac{1}{2}\pi}-Mp(\sin.2L''''-\sin.\ 2\ L'');\end{aligned}\right\}(H),$$

from which it will be easy to determine the values of m and p, which ought to be already very near approximations, since only the quantities of the order α^2 have been neglected.

Let M^0 and p^0 be the first approximating values of M and p; then, to have them more accurate, we may make $M = M^0(1 + x)$, $Mp = M^0 p^0 (1 + y)$, $M\,q = M^0 z$; and, by substituting in the general equation (G) the quantities relative to the four arcs SS$''$, S$''$ S$''''$, SS$'''$, S$'$ S$''''$, we shall have the four following equations, viz.

$$\frac{S''-S}{M^0} = \frac{L''-L}{\frac{1}{2}\pi}(1+x) - p^0(1+y)(\sin.2L''-\sin.2L)$$
$$+ z(\sin.4L''-\sin.4L),$$

$$\frac{S^{iv}-S''}{M^0} = \frac{L^{iv}-L''}{\frac{1}{2}\pi}(1+x) - p^0(1+y)(\sin.2L^{iv}-\sin.2L'')$$
$$+ z(\sin.4L^{iv}-\sin.4L''),$$

$$\frac{S'''-S}{M^0} = \frac{L'''-L}{\frac{1}{2}\pi}(1+x) - p^0(1+y)(\sin.2L'''-\sin.2L)$$
$$+ z(\sin.4L'''-\sin.4L),$$

$$\frac{S^{iv}-S'}{M^0} = \frac{L^{iv}-L'}{\frac{1}{2}\pi}(1+x) - p^0(1+y)(\sin.2L^{iv}-\sin.2L')$$
$$+ z(\sin.4L^{iv}-\sin.4L').$$

But M^0 and p^0 having been determined by the equations (H), it is evident that the first two equations above will be reduced to

$$0 = \frac{L''-L}{\frac{1}{2}\pi}x - y(\sin.2L''-\sin.2L) + z(\sin.4L''-\sin.4L),$$

$$0 = \frac{L^{iv}-L''}{\frac{1}{2}\pi}x - y(\sin.2L^{iv}-\sin.2L'') + z(\sin.4L^{iv}-\sin.4L'');$$

so that we shall have four equations of this form.

$$\left.\begin{aligned} 0 &= f\,x - g\,y + h\,z \\ 0 &= f'\,x - g'\,y + h'\,z \\ e'' &= f''\,x - g''\,y + h''\,z \\ e''' &= f'''\,x - g'''\,y + h'''\,z \end{aligned}\right\} \ldots\ldots (J),$$

from which the values of x, y, and z must be found. In this kind of analysis, of which astronomical questions afford many examples, it is not necessary to attempt to resolve exactly three of the equations, which would cause all the error to be transferred to the fourth; but we must endeavour to compensate the errors in such a manner that they may be nearly equal in each of the four, which will not be attended with any difficulty when the numeral coefficients are substituted.

The value of x being known, we shall immediately have the length of the quarter of the meridian $M = M^0 (1+x)$, which is the chief object of these investigations.—Then the values of y and z will afford valuable notions relative to the figure of the meridian.

We shall first have

$$p = \frac{p^0(1+y)}{1+x)}, \text{ and } q = \frac{z}{1+x}$$

or simply, by performing the divisions,

$$p = p^0 (1 + y - x), \; q = z.$$

But we have made

$$p = \tfrac{3}{4} \frac{m+n - \frac{1}{2} m^2}{\frac{1}{2} \pi}; \; q = \tfrac{15}{16} \frac{m^2 - \frac{1}{2} n}{\frac{1}{2} \pi};$$

Therefore let $\frac{2}{3} p \pi + \frac{16}{15} q \pi = \mu$; we shall have $\mu = \frac{3}{2} m^2 + m$, from which $m = \mu - \frac{3}{2} m^2$; and as m is very small, we may, for the first approximation, make $m = \mu$; then more accurately

$$m = \mu - \tfrac{3}{2} \mu^2;$$

lastly, from the preceding value of q, we obtain

$$n = 2 m^2 - \tfrac{16}{15} q \pi.$$

The quantity $m + n$ is therefore now known. Thus, from the preceeding Note, we have

$$m + n = \alpha' = \frac{1}{\varepsilon}.$$

Therefore the value of the ellipticity of the earth is

$$\alpha = \frac{1}{\epsilon + 1}.$$

Finally, if the curve of the meridian, the equation of which is $v = b\,(1 + m \sin.^2 \varphi + n \sin.^4 \varphi)$,
be an ellipse, it is necessary, as has already been remarked, that $n - \frac{3}{2}\,m^2 = 0$, or, which amounts to the same, that the equation $\frac{1}{2}\,m^2 - \frac{16}{15}\,q\,\pi = 0$, may be fulfilled.

As to the value of the semiconjugate axis b, it may evidently be obtained from the equation

$$M = b\,(1 + \tfrac{1}{2}\,m + \frac{3}{8}\,n + \tfrac{1}{4}\,m^2)\,\tfrac{1}{2}\,\pi;$$

and then, by this formula

$$b = a\,(1 - \alpha),$$

we shall have the value of the equatorial radius a, which has been denoted by ϱ in No. 10.

XIII. It would not be difficult to prove that the normal A M $= r$ (*fig.* 60) has for its expression

$$r = b\,(1 + 2\,m - m\,\cos.^2 \mathrm{L}).$$

Then, by the method followed in No. 10, we shall obtain

$$\mathrm{C\,N} = 2\,m\,r \sin.\ \mathrm{L}',\ \mathrm{M\,N} = m\,r\,\varphi^2 \sin.\ \mathrm{L},$$

$$\text{angle N B M} = m\,\varphi^2 \sin.\ \mathrm{L}.\ \cos.\ \mathrm{L},$$

and it may be concluded, that

$$\mathrm{L}' = \mathrm{L} - \tfrac{1}{2}\,\varphi^2 \operatorname{tang.} \mathrm{L} - m\,\varphi^2 \sin.\ \mathrm{L} \cos.\ \mathrm{L}.$$

Reciprocally, we shall have

$$\mathrm{L} = \mathrm{L}' + \tfrac{1}{2}\,\varphi^2 \operatorname{tang.} \mathrm{L}' + m\,\varphi^2 \sin.\ \mathrm{L}' \cos.\ \mathrm{L}',$$

by preserving the notation in the number referred to.

After demonstrating and applying the formulæ of Legendre, for determining the latitudes and longitudes of places situated upon an elliptic spheroid, those of Delambre, on the same subject, will be indicated in this place, as they are very accurate and very convenient for calculation. The reader is referred for their demonstration to the excellent *Méthodes analytiques pour la Détermination d'un Arc du Meridien.*

The difference of the methods employed by these two geometers, for calculating the respective positions of places on the earth, consists in this: Legendre adopted for the distance between two points, the arc of a great circle comprised between their verticals; while Delambre took for this distance the chord of the same arc. Indeed, the excess of the arc above the chord is often insensible in geodesic operations, even the most delicate; yet, when we aim at great precision, we ought not to neglect it. Now, if the side of a spherical triangle very little curved be denoted by b, for a sphere of which the radius $= 1$, and the corresponding chord by k, we shall have by the first series (A) No. 9,

$$b - 2\sin.\tfrac{1}{2}b = b - k = \frac{b^3}{24}.$$

When B is the length of an arc and K that of its chord, for a radius $= \varrho$, we have evidently

$$b = \frac{B}{\varrho};$$

therefore $b - k$ or $\varepsilon = \frac{1}{24}\frac{B^3}{\varrho^3}$.

Such is the excess of the arc above its chord, on the supposition that b makes part of the radius taken for unity; but that excess will be given in the same measures as the radius of the earth, by multiplying the value of ε by ϱ; that is to say, we shall then have

$$B - K \text{ or } \Sigma = \frac{1}{24}\frac{B^3}{{}^3\varrho}.$$

Lastly, we shall have Σ in seconds by means of the formula $\frac{1}{24}R''\frac{B^3}{\varrho^3}$.

This being premised, let

φ be the arc expressed in seconds, corresponding to the chord K of a terrestrial arc, that is, to a side of a triangle.

L the known latitude of one extremity of the chord K,

L′ the required latitude of the other extremity,

M the known longitude
M′ the required longitude } reckoned from the south towards the west from 0° to 360°.

Z the known azimuth
Z′ the required azimuth } reckoned from the same;

we shall then have

$$\varphi = \frac{R'' K}{\varrho}(1 - \tfrac{1}{2} e^2 \sin.^2 L).$$

$$L' = L - (\varphi \cos. Z + \tfrac{1}{2} \varphi \sin. \varphi \sin.^2 Z \operatorname{tang.} L)$$
$$(1 + e^2 \cos.^2 L).$$

$$Z' = 2\,q + Z - \varphi \sin. Z \operatorname{tang.} L' - \tfrac{1}{4} \varphi \sin. \varphi \sin. 2\,Z.$$

$$M' = M + \varphi \frac{\sin. Z}{\cos. L'}.$$

By making $Z = q$ in the last three equations, and $\sin. \varphi = \varphi$, the first terms of the analogous formulæ, No. 10, will be found again.

Measure of a Base.

Ground cannot be every where found which is convenient for measuring long bases. In fact, we rarely meet, even in the finest plains, with an extent in a right line of several leagues, without obstacles, or without sensible inequalities, which prevent the two extremities of that right line from being seen from each other.

The longest bases that have hereto been measured did not generally exceed thirteen or fifteen thousand English yards, or thereabouts; that which has lately been measured in Bavaria was 11107 toises, about 23669 English yards, but this length can seldom be attained. Besides,

great length alone will not suffice for bases, and it is also requisite that they should be properly connected with two summits, at least, of the chain of triangles.

Indeed it is to be wished that bases were always connected with the summits of the chain by equilateral triangles; but this cannot generally be done; wherefore we must confine ourselves to making them as long as possible, and be satisfied with isosceles triangles, or such as are nearly so.

This shows the conditions to be fulfilled in the choice of bases; the most convenient spots for them are generally found on the borders of the sea, on the banks of rivers which have but little declivity, across passable marshes, or along roads; obstacles frequently oblige us to deviate a little from the right line; then the base is formed of two lines which make an angle very nearly equal to 180°, and in this case it ought to be reduced to a right line.

The methods of measuring long bases are various; sometimes they are measured on the ground, or on a kind of bridge elevated above it; sometimes, too, a horizontal line is followed, or else, the inclination of the ground when it is uniform; in which case, this inclination must be reduced to the horizontal line. The latter method, when it can be employed, is preferable to the former, because the small errors which necessarily result from the change of level are avoided.

In all cases, convenience and exactness should be sought; for, this operation indispensably requires such an accuracy as will answer to that given by the instrument which is used for measuring the angles.

In order to measure a base on the ground, the surface should be smoothed and levelled, which is not always practicable. To trace the base, poles are fixed in the

ground at proper distances from each other; the best instrument that can be employed for placing them in the requisite direction is a *Transit* instrument, but others may be used. These poles are not suffered to remain during the whole measurement, as they would run a risk of being carried away, but piquets are driven into the ground at the places where they stand, and poles are substituted afterwards for these piquets, as we advance in the operation.

Rods of various materials have been employed for measuring bases: the English used glass rods for the measure of the base on Hownslow Heath. Rods of deal, of iron, and of platina, were employed for measuring the two bases which served to determine the arc of the meridian by which the length of the *metre* was fixed.

Deal rods appear to be the least exact, on account of their hygrometrical variations. But if the precaution be taken to boil them for a long time in some oilly matter, and then to cover them with a thick coat of paint, the different states of humidity of the atmosphere will not affect them, and they will not only become as good as metal rods, but prove much more convenient, as they are lighter. It may be added, that they expand much less by heat than metal rods, wherefore they are preferable to them in that respect. They should be properly supported so that they may not get bent.

Three, four, or five rods are generally used, and their extremities may either be brought into contact, or not suffered to touch each other, as has been lately done in France, with a view to avoid the recoil that may be produced by bringing one rod into contact with another already placed. But then, it will be necessary to measure with a vernier the space left between them; besides, it is scarcely possible that any recoil may take place, if

one rod is brought to press gently against three or four which afford a considerable resistance.

The most important operation, perhaps, is to determine exactly the total length of the rods in common measures when they are placed end to end, which the French call *portées:* for this purpose, a standard may be constructed on the ground, as has been done in Bavaria.

Great care should be taken in measuring bases, not to commit any mistake in reckoning the *portées;* for this, a line or chord is used, which is made equal to a certain number of *portées*, and serves for placing gradually, as we advance, poles which indicate the number of *portées* that have been passed over.

When rods of wood or metal are used, it is necessary to take an account of the variations of the thermometer, as it will afford the means of bringing the whole length of the base to the same temperature.

END OF THE FIRST PART.

PART II.

PORTATIVE BAROMETRICAL TABLES,

Giving the Difference of Level by a Simple Subtraction; with an Introduction, containing the History of the Barometrical Formula, and its Complete Demonstration by the Simple Elements of Algebra.

BY M. BIOT.

TRANSLATED AND ADAPTED TO ENGLISH INSTRUMENTS AND DIVISIONS.

PRELIMINARY NOTICE.

About one hundred and sixty years have elapsed since Pascal, having caused Torricelli's barometer to be carried to the top of Puy-de-Dome, remarked that this instrument presented the means of levelling the most distant places.

This first idea was not abandoned, but in order to render it applicable for ascertaining the lengths of the columns of air passed through, according to the diminutions observed in the column of mercury, many elements still remained to be determined. We had the balance, but knew not the values of the weights. The first thing that it was necessary to know was the law of the condensation of air under different pressures. Mariotte in France, Boyle and Townley in England, found from experiment that the density of this fluid is

proportional to the compressing weight. This law is true only when the temperature of the air remains constant; but attention was not then paid to this important restriction, which, in fact, could not be indicated by experiments in which the compared volumes of air always had nearly the same temperature. The law of the compression of air being known, Halley made use of it for calculating the decrease of density in the beds of the atmosphere at different heights; and he thus discovered the mathematical formula, by means of which the difference of altitude of two stations may be calculated from the heights of the mercury in the barometer, observed at each of them. Newton, in his Principia, perfected that theory by showing what regard was to be paid to the diminution of the gravity of the molecules of air, according as the distance from the surface of the earth increased. But, what is very remarkable in so scrupulous an observer of nature, he omitted also to consider the effect of the variations of heat, and of the progressive decrease of the temperature on the density of the beds of air. At this time observations of the barometer and thermometer were not even employed in the measure of astronomical refractions. It was Bradley, Mayer, and Lacaille, who began to introduce these corrections about 1750; until then, the only method was to use different tables of refraction for summer and winter.

The barometrical formula, without the correction which renders it applicable at all temperatures, could only furnish a very imperfect approximation. Thus, the philosophers and astronomers who endeavoured to apply it, found that it succeeded only in a few instances, and that, in general, it was subject to considerable errors which did not appear to follow any law. Different hypothesis were formed by some for explaning these irregu-

larities, and others concluded that the formula was absolutely to be rejected. No person thought of the true cause; and this omission is very extraordinary when we reflect that Bouguer and Lambert, two men of the most singular merit and different talents, the one an ingenious and accurate philosopher, the other a most inventive and acute geometer, were both much occupied with the barometer and its application.

It was M. Deluc who at last discovered the source of all these anomalies; he sought in the observations themselves the correspondence between the temperature of the air and the corrections which the formula required. Numerous experiments on the comparative expansions of air and mercury enabled him to perceive the law that those corrections ought to follow, and the intensity which should be assigned to them.

This remarkable discovery, by giving to the barometrical formula an unexpected accuracy, animated the zeal of philosophers, and barometrical observations were multiplied. Dr. Maskelyne undertook to translate the new formula into English measures. Mr. Playfair added a correction for the variation of gravity in different latitudes. Sir George Shuckburgh, by very exact measures, verified the results of M. Deluc, and gave them a greater degree of precision. General Roy also made an application of it at a great number of places in the Britannic Isles. The Alps were levelled by M. M. Saussure and Pictet; the Pyrenees, by M. Ramond, and the Andes, by M. Humboldt. The barometer, rendered portative, became an indispensable instrument to all well-informed travellers.

However, notwithstanding so many successful applications, the theory of barometrical levelling was far from being brought to its most simple terms. M. Deluc had

adapted the constant co-efficients of his formula to a certain temperature, which he called the *normal temperature*, and which he had fixed from the condition, that, for this temperature, the difference of level became a decimal multiple of the difference of the tabular logarithms of the observed barometrical heights. All the corrections relative to temperature which the formula required, commenced therefore, according to M. Deluc, at the normal temperature; in consequence of which this point of commencement changed whenever the formula was applied to any other measures than French toises. These variations were very inconvenient, and it appeared much more natural to make all the corrections commence at some fixed term, as the freezing point, which is given by experience, and common to observers of all countries. This is what Laplace has done in a chapter of his *Mécanique céleste*, in which he has established the barometrical formula upon the most simple and accurate data. He determines the correction for temperature, relative to the expansion of air, according to the experiments of M. Gay-Lussac; but he has modified his results in such a manner, as to take into the account the humidity of the atmosphere; and, what is very fortunate, the sum of this correction and the coefficient of the expansion of air is just equal to $\frac{4}{1000}$. With respect to the expansion of mercury, Laplace employed the values he obtained in conjunction with the illustrious Lavoisier, in experiments on the expansion of bodies, of which there unhappily remains only a small number of results. Finally, he determined the general coefficient of the formula from barometrical observations themselves, by combining for this purpose a great number of experiments made in the Pyrenees by M. Ramond, with a care and an accuracy that

leave nothing to desire. The value of this coefficient has since been confirmed in a direct manner by the experiments made by M. Arago and myself, on the comparative weights of air and mercury; so that all the elements of the barometrical formula, the research of which has cost travellers so much labour, might have been obtained directly, and with as much accuracy, without going out of the chemical laboratory. Laplace's formula, founded upon data so exact and so ably combined, represents observations better than any other in which these advantages are not united. The rigorous proofs to which M. M. Ramond and Daubusson have submitted it experimentally, proved its utility. It still remained, however, to render the observations comparable with each other, though made with different barometers. But Laplace has shown that the different indications of these instruments, in circumstances otherwise equal, are the effects of cappillary action, and he has given tables for correcting this effect.

The barometrical formula being thus improved, or rather perfected, observations with the barometer have been considerably multiplied, and carried to a degree of precision almost incredible, as the observers were guided by accurate rules; we may now entertain a hope, that in the space of some years the general levelling of Europe will be obtained; and we may undertake, agreeable to the idea suggested by Laplace, to add to the latitude and longitude of cities their height above the level of the sea, as a third co-ordinate which would completely determine their positions.

In order to favour this idea, I have inserted, in the first edition of my astronomy, tables by which we might find directly the height of a place above the level of the sea, on the supposition that, from a great number of observa-

tions made at the same place, the mean heights of the barometer and thermometer had been determined. I have had the satisfaction of knowing that these tables have been useful to many observers.

Since that time, an astronomer known by his accuracy, M. Lindenau, has published barometrical tables on a much more extended plan. They show the difference of level of any two stations whatever, where the corresponding heights of the barometer and thermometer have been observed. But, though these tables are very useful for calculating barometrical observations in the study, the space they occupy, which is 170 pages, does not permit them to be conveniently carried in mountainous journeys; and still less to be used at the places themselves. Horsley and Schuckburg have also long since attempted to calculate tables from the formula of Deluc; but their work, inserted in the Philosophical Translations, wants one essential quality, which is, simplicity. Complicated tables can be consulted only by persons accustomed to calculation, and for these the best of all tables is the formula itself.

M. Oltmans, a young astronomer who has made himself known by a rare and singular devotedness to numerical calculations, having undertaken to calculate, by Laplace's formula, the barometrical observations which M. Humboldt had made in his celebrated travels between the tropics, was naturally led by this great undertaking to desire tables for abridging it. He constructed some which are very remarkable for their simplicity, without causing the formula to lose any of its rigour; he has succeeded in bringing it into three tables of a double entry, which are contained in only fourteen folio pages.

The height is calculated by simple additions and subtractions. But it is necessary to enter the tables four

times with different arguments which are deduced successively one from another; this requires an attention always very difficult to an observer who is not accustomed to the use of astronomical tables. It is besides very inconvenient to French observers, as these tables suppose the height of the barometer to be observed in lines, and give the difference of level in toises. This construction necessarily requires troublesome reductions when the observations are made in parts of the metrical scale, which is almost always the case at present in the French barometers.

The publication of a second edition of my astronomy having made it a duty that I should investigate with great care the most simple and accurate methods for all kinds of application which fall within the object of that treatise, I have endeavoured to perfect my former barometrical tables, and to render their utility more general. I was above all desirous of rendering them so simple and commodious, that philosophers, naturalists, and all well-informed travellers might carry them in their journeys, and make use of them upon the spot for calculating their observations. I hope I have succeeded in this attempt, but experience alone can decide.

I have considered first, that, for persons little accustomed to the use of tables, a very short and simple arithmetical computation was preferable to the use of several tables which it would be necessary to consult with different arguments. I have, therefore, thrown into this form the correction for the expansion of mercury, and that for the latitude. The first requires only the observed barometer at the colder station to be reduced to the temperature of the warmer, by adding to the barometrical column $\frac{1}{9742}$ part of its length, for every degree

of Fahrenheit's thermometer. The correction for the latitude is nothing at the parallel of 45°; it will be almost always insensible in observations made in Europe, and naturalists have not any occasion to take it into the account in their travels. But, as it may become necessary to have respect to it in very exact observations, I have inserted it in a small table of fractions of the height which must be added according to the different latitudes.

Suppose, now, that the heights of the barometer and thermometer had been observed at two stations unequally elevated, the calculation of the difference of level would be reduced to two very simple and constantly uniform operations.

With the sum of the temperatures and the height of the barometer observed at the upper station, the table is to be entered, and the number corresponding to this given quantity taken.

With the same sum of the temperatures, and the observed height of the barometer at the lower station, the same table is to be entered again, and the corresponding number taken. The difference of these numbers is the difference of level. If this difference be greater than 2180 yards, there is another quantity to be added, which is found in the same table, and always by the same proceeding. This suffices as far as 4360 yards; but the use of the table may be extended indefinitely by a similar proceeding, and the calculation never becomes more complicated.

Such is the form of the portable barometrical tables, which are here presented to well-informed travellers, that they may be easily carried in the pocket, and the observations thereby calculated on the spot at the same time they are made.

These tables are preceded by instructions explaining

their use, and containing a complete demonstration of the barometrical formula. This demonstration only requires the reader to be acquainted with the first rudiments of Algebra, but persons who are entirely strangers to these notions, may pass immediately to the numerical examples inserted at the end of this part of the Treatise, with specimens of calculation for all cases, which may be constantly taken as models, no alteration being ever requisite.

All my efforts have been used to bring these tables to the greatest degree of simplicity of which they are susceptible; and I shall esteem myself very happy if they should increase the taste for barometrical observations, and contribute to the perfection of physical geography, too little cultivated among us.

On the Measure of Altitudes by barometrical Observations.

As the measure of altitudes by barometrical observations may be of very frequent utility, I have united in this chapter all the details that can be desired, relative to both the demonstration of the rigorous formula and its application.

Conceive a vertical tube to be filled with air, which extends from the surface of the earth to the limits of the atmosphere. Besides, for the purpose of simplifying the problem, let it be supposed, at first, that the column of air thus formed is composed of air perfectly dry, and the temperature the same throughout its extent. Let it be supposed, also, that the decrease of gravity, according to the greater elevation, is abstracted; so that this force may be considered as acting with an equal intensity at all heights. On these suppositions, let us examine the

the state of equilibrium of the column. It is evident that each molecule will be compressed by the weight of all those above it; and as the air, in consequence of its elasticity, is condensed proportionally to the compressing force, its density will decrease from the bottom to the top by an insensible gradation. To discover the law of this gradation, conceive the column to be divided into an indefinite number of very thin beds or strata, the thickness of each not exceeding a thousandth part of a yard for example; so that the density may be sensibly the same throughout the height of the same stratum, and vary only from one to another. Then, if the barometer be carried successively into each of these beds, at different distances from the centre of the earth, there will be a certain ratio between these distances, represented by x_1, x_2, x_3, and the elevations of the mercury in the barometer denoted by H_1, H_2, H_3. This is the ratio which it is required to determine.

For this purpose, it may be remarked, that the thickness of the first bed or stratum is expressed by $x_2 - x_1$. The depression of the mercury, in raising it through that bed is $H_1 - H_2$. Consequently, at that elevation, a column of air of the height $x_2 - x_1$, weighs as much as a column of mercury of the same base, and having for its height $H_1 - H_2$. Thus the density of that stratum, compared with that of mercury, is $\frac{H_1 - H_2}{x_2 - x_1}$; for the densities are reciprocally as the magnitudes when the weights are equal.

But this ratio, between the density of the stratum and that of mercury, may be obtained in a different manner. For, at an equal temperature, the density of each stratum is proportional to the pressure it sustains, that is, to the weight of the superior beds. Now, since all the beds are

supposed to have the same temperature, the pressure which each sustains is proportional to the height of the mercury in the barometer. Thus, on the suppositions we have admitted, the density of the different beds may be represented by CH_1, CH_2, CH_3,..C being a constant co-efficient, common to the whole column. In this manner, two expressions for the density of the first stratum are obtained, viz. $\frac{H_1 - H_2}{x_2 - x_1}$ and CH_1; and by putting these equal to each other we shall have

$$CH_1 = \frac{H_1 - H_2}{x^2 - x_1},$$

from which we obtain $H_2 = H_1 \left\{ 1 - C(x_2 - x_1) \right\}$.

The same relation will subsist in passing from the second stratum to the third, from the third to the fourth, and so on in successsion, at least on the suppositions that have been admitted; so that we shall have the following equations.

$$H_2 = H_1 \left\{ 1 - C(x_2 - x_1) \right\},$$
$$H_3 = H_2 \left\{ 1 - C(x_3 - x_2) \right\},$$
$$H_4 = H_3 \left\{ 1 - C(x_4 - x_3) \right\},$$
$$H_5 = H_4 \left\{ 1 - C(x_5 - x_4) \right\},$$
&c.

Or, if we represent by D the thickness of the stratum, which is supposed to be always the same,

$$H_2 = H_1 (1 - CD),$$
$$H_3 = H_2 (1 - CD),$$
$$H_4 = H_3 (1 - CD),$$
$$H_5 = H_4 (1 - CD),$$

from which we obtain the following values:

$$H_2 = H_1 (1 - C D),$$
$$H_3 = H_1 (1 - C D)^2,$$
$$H_4 = H_1 (1 - C D)^3,$$
$$H_5 = H_1 (1 - C D)^4,$$

and we shall have between the differences of level and the depression of the mercury the corresponding series:

$$x_2 - x_1 = D \qquad \frac{H_2}{H_1} = (1 - C D),$$

$$x_3 - x_1 = 2D \qquad \frac{H_3}{H_1} = (1 - C D)^2,$$

$$x_4 - x_1 = 3D \qquad \frac{H_4}{H_1} = (1 - C D)^3,$$

$$x_5 - x_1 = 4D \qquad \frac{H_5}{H_1} = (1 - C D)^4,$$

the quantity $1 - C D$ is necessarily a fraction; for C and D are both positive; and let C be what it may, D may always be taken sufficiently small, that the product C D may be a fraction. Then the different powers of $1 - CD$ will be less and less. Thus it appears from the preceding series, that, *when the altitudes above the first station increase in an arithmetical progression, the heights of the mercury in the barometer decrease in a geometrical progression.*

In order to obtain this result, it has been supposed that each stratum of air, of a thousandth of a yard in height, was throughout of an equal density. This supposition is not strictly true; but it approaches more and more to the truth as the thickness of the beds, or strata, is less. Now, instead of taking a thousandth of a yard for the thickness, we may take it a hundredth part of that, or any other smaller dimension, and the error will be diminished indefinitely, without our being prevented from arriving at the

same consequences. Thus, the law which we have found is true in itself, and independent of all suppositions relative to the thickness of the beds, or strata. This is what the following calculation confirms.

If the rank of any term whatever in the two preceding series be denoted by n, and its value found, which may be done in the second series by means of logarithms, we find

$$n = \frac{x_{n+1} - x_1}{D}, \quad n = -\frac{(\log. H_1 - \log. H_{n+1})}{\log. (1 - C D)},$$

from which we obtain

$$x_{n+1} - x_1 = -\frac{D (\log. H_1 - \log. H_{n+1})}{\log. (1 - C D)}.$$

$x_{n+1} - x_1$, is the difference of level of the two stations; for greater simplicity, it may be denoted by X. H_1 is the height of the mercury answering to the lower station, which may be represented by H. Lastly, H_{n+1} is the height of the mercury at the upper station, which shall be denoted by h; for, having nothing to consider but the extremities of the column, the accents by which the different beds were distinguished, are not of any further utility. We shall then have

$$X = \frac{-D}{\log. (1 - C D)} (\log. H - \log. h).$$

The value of X seems to depend upon the thickness D, which we have supposed to be that of the different beds of air; but, in fact, it does not depend upon this quantity. For, by developing the logarithm of $1 - C$ D, we have $\log. (1 - C D) = -\frac{1}{M}\Big(C D + \frac{C^2 D^2}{2} + \frac{C^3 D^3}{3}$

$+ \frac{C^4 D^4}{4} + \&c. \Big)$

M being the modulus of the common tables, or 2.302585092994, consequently

$$-\frac{D}{\log.(1-CD)} = \frac{M}{C + \frac{C^2 D}{2} + \frac{C^3 D^2}{3} + \&c.}$$

The thickness D is supposed extremely small; and to attain the utmost rigour, it must be made equal to nothing, which gives $-\frac{D}{\log.(1-CD)} = \frac{M}{C}$; then this co-efficient becomes independent of D; and it now appears, that, by deferring till this time, to suppose that quantity equal to nothing, it was only to effect the possibility of establishing the reasoning, and performing the calculus.

From this result, we shall have the formula

$$X = \frac{M}{C} (\log. H - \log. h),$$

that is to say, the difference of level is proportional to the difference of the logarithms of the heights of mercury in the barometer.

There remains only the co-efficient C to find. Now, by representing the density of the air under the pressure H by δ, that of mercury being unity, we have, agreeable to the preceding suppositions, $\delta = CH$, H being the height of the mercury in the barometer. The value of C may therefore be obtained, if we have, from very accurate experiments, the ratio of the densities of air and mercury under a given pressure of the atmosphere.

This ratio is not the same in all countries; for in all countries the force of gravity has not the same intensity, as is proved by experiments with the pendulum; and the ratio $\frac{\delta}{H}$ varies with the force of gravity. In fact, δ is the density of air under a given pressure; for example, under a pressure of 0.76 of a metre, or .831136 of an English yard. But, according as the intensity of gravity

is greater or less, a column of mercury having always .831136 of a yard of altitude, will weigh more or less; consequently, the air sustaining this pressure will be more or less compressed. Now, from experiments with the pendulum made in different latitudes, it is found, that, by calling 1 the force of gravity at the parallel of 45°, the gravity under any other parallel of latitude ψ, is expressed by $1 - 0.002837 \cos. 2\psi$. The density δ, being proportional to the force of gravity, will vary in the same ratio, that is, by calling it δ at the parallel of 45°, and under the pressure H, it will become for any other latitude, and under a column of mercury of the same length, $\delta(1 - 0.002837. \cos. 2\psi)$.

The co-efficient C, which expresses the ratio of the density to the height of the barometrical column, ought therefore to vary in the same proportion, and consequently it will become $C(1 - 0.002837. \cos. 2\psi)$, which being substituted in the value of X, gives

$$X = \frac{M}{C(1 - 0.002837. \cos. 2\psi)} \log. \left(\frac{H}{h}\right);$$

in this manner it will be sufficient to find the co-efficient $\frac{M}{C(1 - 0.002837)}$ by experiment for a given latitude. For then ψ being known, we shall also know $\frac{M}{C}$; and the formula will become applicable to all possible latitudes.

This formula may be rendered more convenient by making the denominator disappear, which is easily done; for the fraction $\frac{1}{1 - 0.002837. \cos. 2\psi}$ being transformed into a series by division, becomes $1 + 0.002837. \cos. 2\psi + 0.00000804857. \cos.^2 2\psi \ldots + \&c.$ or simply $1 + 0.002837. \cos. 2\psi$, by taking only the first

term, which alone is sensible. We shall then have $X = \frac{M}{C}(1 + 0.002837 . \cos. 2\psi) \log. \left(\frac{H}{h}\right)$.

Hitherto it has been supposed that the value of the co-efficient C, or $\frac{\delta}{H}$, was the same in all the strata of the column. But this is not so in nature, and several causes concur in making this ratio vary. The principal of these is the difference of temperature of the strata. For the elasticity of air is increased by heat; so that with a less density, it may sustain an equal column of mercury, which causes the ratio $\frac{\delta}{H}$, or C, to vary. This ratio also varies according to the quantity of aqueous vapour that is suspended in the different strata of the column. For this vapour weighs less than dry air of an equal elastic force; so that its introduction into the different beds of air renders them similarly susceptible of sustaining, with a less density, a column of mercury of equal height. Lastly, the decrease of gravity, arising from greater elevation, is also another cause of change; for, in consequence of this decrease, a column of mercury, the length of which is H, weighs less as it is more distant from the centre of the earth; now, if it weigh less, the beds of air into which it is transported are less compressed; thus the ratio of their density to the length of the mercurial column, or $\frac{\delta}{H}$, is no longer the same for these beds as for those below them. Let us now search the numerical value of the influence of these various causes on the co-efficient C.

We shall commence with the decrease of gravity in a vertical direction. Let g_1, g_2, g_3, be the different intensities of that force in the different beds, or strata. The

weight of the columns of mercury H_1, H_2, H_3, which they solicit, are proportional to them; consequently, if all other circumstances were equal, the densities of the beds of air compressed by these columns would also have the same proportion. The ratio $\frac{\delta}{H}$, or C, ought therefore to vary from one bed to another proportionally to the gravity g.

Let us now consider the influence of temperature. In virtue of this cause, a mass of air, the magnitude of which will be 1 at zero of temperature, becomes at t degrees above the freezing point of Fahrenheit's thermometer $1 + 0.0020833\,t$, the barometrical pressure remaining the same. Now, the densities of this mass under a constant pressure are reciprocal to the spaces which it is made to occupy; consequently, if its density at zero be 1, its density at t degrees will be $\frac{1}{1 + 0.0020833\,t}$, the pressure remaining the same. The ratio $\frac{\delta}{H}$, or C, ought therefore to vary in the different beds proportionally to $\frac{1}{1 + 0.0020833\,t}$.

We shall now examine the influence of aqueous vapour. According to the experiments of Saussure and Watt, the weight of this vapour is to that of air as 10 to 14, when their temperatures and elastic forces are the same; that is, when the air and vapour, having the same temperature, sustain equal columns of mercury. Hence, the substitution of this vapour in the strata of air renders them specifically lighter without diminishing their elasticity. To find the value of this effect, let H be the barometrical pressure which a certain bed of air supports: call F the elastic force of the aqueous vapour con-

tained in it, that is, the part of the barometrical pressure which the vapour sustains. The total weight of the bed may be considered as composed of two parts; *viz.* of a certain quantity of vapour, the elastic force of which is F, and of a certain quantity of dry atmospheric air, the elasticity of which is $H - F$; let p be the whole weight of the bed, if it were entirely composed of dry air, under a pressure H. The weight of the same volumn of dry air, under a pressure equal to $H - F$ will be $p\frac{(H-F)}{H}$. The weight of the same volume under the pressure F, will be $\frac{p\,F}{H}$; so that, if this volume, remaining always under the pressure F, were composed entirely of aqueous vapour, its weight would be $\frac{10}{14}$ of the preceding, that is, $\frac{10}{14}.\frac{p\,F}{H}$. Now, it is known by very positive experiments, that, in a mixture of vapour and air in a state of permanent equilibrium, these two fluids are uniformly expanded throughout the whole space which they occupy. Thus, the weight of the mixture in the preceding proportions, will be equal to the sum of the weights of air and vapour which occupy the given space under the pressures $H - F$ and F; that is, the weight will be $p.\frac{(H-F)}{H} + \frac{10}{14}.\frac{p\,F}{H}$, or simply $p.\frac{(H-\frac{2}{7}F)}{H}$. But, before the introduction of the vapour, the weight of the same volume of dry air, sustaining the same pressure H, was denoted by p. The densities being proportional to the weights, if δ represent the density of a stratum in the dry state, its density in the humid state will be $\delta.\frac{(H-\frac{2}{7}F)}{H}$, or $\delta.\left(1-\frac{2}{7}\frac{F}{H}\right)$,

the pressure being the same. From this it appears, that the introduction of aqueous vapour into the beds of air causes the ratio $\frac{\delta}{H}$ or C to vary proportionally to $1 - \frac{2}{7}\frac{F}{H}$.

By resuming the three kinds of variation to which the co-efficient is subject, the most general expression for it will take the following form:

$$C = \frac{Ag\left(1 - \frac{2}{7}\,\frac{F}{H}\right)}{1 + 0.0020833\,t},$$

A being a constant quantity common to all the strata. There only remains to substitute in this expression for g, H, F, and t, their values relative to the different beds.

The value of the factor g shall first be calculated. It is known, that, in receding from the centre of the earth, the force of gravity is reciprocally as the square of the distance. The distances of the different beds have been denoted by x_1, x_2, x_3, therefore, by calling g_1, g_2, g_3, the corresponding intensities of gravity, we shall have

$$g_1 = g_1, \quad g_2 = \frac{g_1 x_1^2}{x_2^2}, \quad g_3 = \frac{g_1 x_1^2}{x_3^2}, \quad \&c.$$

We shall now find the term dependent on the aqueous vapour. The tension F of that vapour is always very little in the temperatures in which barometrical observations are generally made. By calculating their values in parts of a yard, for the point of extreme saturation, according to the formulæ which Laplace has given in his *Mécanique céleste*, and which he deduced from the experiments of Dalton, we find

$$\left.\begin{array}{l}\text{at } 32^\circ \text{ of Fahrenheit's thermometer} \ldots F = 0.00560142 \\ \text{at } 86^\circ (\text{or } 54^\circ \text{ above the freezing point}) \ldots F = 0.0346562\end{array}\right\}\text{yds.}$$

and between these two limits which are nearly those of barometrical observations, the increase of F may be represented with sufficient accuracy by the arithmetical

progression $F = \overset{\text{yd.}}{0}.00560142 + \overset{\text{yd.}}{0}.00052547\,t$, *t* being the number of degrees in the temperature above the freezing point in Fahrenheit's thermometer. Though this formula may not be very exact, yet it will be sufficient in this case, because of the small influence which it has on the observed heights. But, before it be applied to the atmosphere, there must still be a modification made. It relates to the point of extreme saturation, which very rarely takes place in the atmosphere; and consequently, the value which it gives for F will be almost always much too great. It is true that we cannot determine any thing certain relative to the quantity of aqueous vapour suspended in the atmosphere. This quantity is extremely variable at different times; nay, it varies from one stratum to another in a very irregular, and sometimes abrupt manner, as is experienced on mountains where strata, very little charged with vapours, succeed to others which are at a *maximum* of humidity. But, independent of these extraordinary circumstances, there is the greatest reason to believe that we shall approach the most frequently to nature by avoiding the extremes, and then the most simple method is to take for the expression of F in the atmosphere, the half of the value which answers to the point of extreme humidity; that is, $F = \overset{\text{yd.}}{0}.00280071 + \overset{\text{yd.}}{0}.00026274\,t$.

On substituting this value in the expression for the co-efficient C, it must be multiplied by the variable factor $\frac{2}{7\,H}$. But, because of the smallness of this correction, and the little difference in the values of H in the extent of the columns of air which are usually measured, it will be sufficient to put for H the constant value .831136 yd. which is the mean pressure at the level of the sea. This substitution will have even the advantage of diminishing

the correction for humidity in the upper strata of the column, which agrees with nature; for the humidity of these strata generally decreases as their height is greater, and sometimes the most elevated possess a great degree of driness. By adopting this simplification, we shall have $1 - \frac{2\,F}{7\,H} = 1 - \frac{2}{7 \times .831136\,\text{yd.}}\ (\overset{\text{yd.}}{0}.00280071$ $+\ \overset{\text{yd.}}{0}.00026274\,t,) = 1 - \overset{\text{yd.}}{0}.0009628 - \overset{\text{yd.}}{0}.0000903\,t.$ This expression may, without sensible error, be brought to the following form, $(1 - \overset{\text{yd.}}{0}.0009628.)\,(1 - \overset{\text{yd.}}{0}.0000904\,t)$ which gives

$$C = \frac{A\,(1 - \overset{\text{yd.}}{0}.0009628),\ g(1 - \overset{\text{yd.}}{0}.0000904\,t}{1 + 0.0020833\,t};$$

In this manner C contains a constant factor, common to all the strata. The other factor depending upon t, which is still found in the numerator, may be united to that which arises from the temperature. In fact, because of the smallness of the co-efficient $\overset{\text{yd.}}{0}.0000904$, we may, without sensible error, substitute $\frac{1}{1 + \overset{\text{yd.}}{0}.0000904\,t}$ for $1 - \overset{\text{yd.}}{0}.0000904\,t$, then we have in the denominator the product $(1 + \overset{\text{yd.}}{0}.0000904\,t.)\ (1 + \overset{\text{yd.}}{0}.0020833\,t)$. In performing the multiplication, the product of $\overset{\text{yd.}}{0}.0000904$ by $\overset{\text{yd.}}{0}.0020833$ may be neglected. The denominator will then become $(1 + \overset{\text{yd.}}{0}.0021737\,t)$. The co-efficient of t in this result may be changed into 0.0022, without incurring any sensible error, which will simplify the computation. We shall therefore have

$$C = \frac{A\,(1 - \overset{\text{yd.}}{0}.0009628)\,g}{(1 + \overset{\text{yd.}}{0}.0022\,t}.$$

It appears that the consideration of the humidity of the air only increases a little the co-efficient of the expansion which agrees with dry air. A simple letter might have been substituted for the product of the two constant factors which are contained in the numerator; but it has been thought preferable to retain them, in order to show the effect of the humidity upon the co-efficient.

Let us now search, in this general case, the ratio of the heights of the barometer with the elevations of the strata. For this we must recur to the same source of reasonings which we have used in the simple case, at first treated. In considering the first stratum, it has been remarked, that, at this elevation, a column of air, the thickness of which is $x_2, - x_1$, weighs as much as a column of mercury of the same base, and of which the height was $H_1 - H_2$, and it has been concluded, that $\frac{H_1 - H_2}{x_2 - x_1}$, was the ratio of the densities of *mercury* and *air* in this stratum. This consideration is also applicable to the actual case; only as gravity is supposed to vary from one stratum to another, the intensity of this force upon the column of mercury H_2, which takes place in the second stratum, is different from that which solicits the column of mercury H_1. To express the weight of the first stratum of air in parts of the column of mercury H_1, it will be necessary to bring the column H_2, to what it would be if the same gravity g_1 acted upon it; that is, to multiply it by $\frac{g_2}{g_1}$, the ratio of gravity in the two strata. We shall thus have $H_1, - \frac{H_2 g_2}{g_1}$ for the diminution of the barometrical pressure in the extent of the first stratum of air, the thickness of which will always be $x_2 - x_1$ as before

The ratio of the densities of air and mercury in this stratum will therefore be equal to

$$\frac{H_1 - \frac{H_2 g_2}{g_1}}{x_2 - x_1} \quad \text{or} \quad \frac{H_1 g_1 - H_2 g_2}{g_1 (x_2 - x_1)},$$

but this same ratio may yet be expressed by C_1, H_1, by representing by C_1, the value of the co-efficient C in the stratum under consideration; then, by equating these two values, and always denoting the thickness of the stratum by D, we shall have

$$x_2 - x_1 = D, \ \frac{H_1 g_1 - H_2 g_2}{g_1 (x_2 - x_1)} = C_1 H_1;$$

or else by obtaining the value of $H_2 g_2$, $x_2 - x_1 = D$, $H_2 g_2 = H_1 g_1 \left\{1 - C_1 (x_2 - x_1)\right\}$.

The passage from the second stratum to the third, and from the third to the fourth, and so on, will all give similar equations; from these we shall have

$$x_3 - x_2 = D, \ H_3 g_3 = H_2 g_2 \left\{1 - C_2 (x_3 - x_2)\right\},$$

$$x_4 - x_3 = D, \ H_4 g_4 = H_3 g_3 \left\{1 - C_3 (x_4 - x_3)\right\},$$

&c. &c.

By successively performing the eliminations, as in page 178, as far as the last stratum, the rank of which is represented by $n+1$, we shall find $x_{n+1} - x_1 = nD$, $H_{n+1} g_{n+1} = H_1 g_1 (1 - C_1 D)(1 - C_2 D)(1 - C_3 D) \cdots\cdots (1 - C_n D)$.

The second member of this latter equation has as many factors as there are strata. In the first case which we have considered, all these factors were equal to each other; instead of which, in this place they are different, because of the variable nature of C. Now, if this last equation be put into logarithms, we shall have

$$\log. \frac{H_1}{H_{n+1}} + \log. \frac{g_1}{g_{n+1}} = -\log. (1 - C_1 D) - \log.$$

$(1-C_2 D)\ldots\ldots-\log.(1-C_n D)$. Here n is not found explicitly in the second member; we cannot, therefore, eliminate it as at page 178. But we may follow the same method of calculation; and, by taking the value of $x_{n+1}-x$, we find

$$x_{n+1}-x=\frac{-nD\left(\log.\frac{H_1}{H_{n+1}}+\log.\frac{g_1}{g_{n+1}}\right)}{\log.(1-C_1 D)+\log(1-C_2 D)\ldots+\log.(1-C_n D)}.$$

Now, by developing the logarithms, performing the division by D, and then supposing D to be equal to nothing, precisely the same as in the case which was first treated, page 178, we shall obtain

$$x_{n+1}-x=Mn\frac{\left(\log.\frac{H_1}{H_{n+1}}+\log.\frac{g_1}{g_{n+1}}\right)}{C_1+C_2\ldots\ldots\ldots+C_n},$$

M being the modulus of the logarithmic tables, or 2.30258509. This formula is analogous to that at page 179, only, instead of having C in the denominator, we have the sum of all the co-efficients C_1, C_2, C_3, and this is the reason why n is still retained in the numerator. If we were to suppose all these co-efficients equal to each other, and to C, their sum would become nC; n would disappear, and we should obtain identically the first formula.

For the sake of abridgment, the sum of all the co-efficients C_1, C_2, C_3, &c. taken in the whole length of the column of air, may be denoted by SC_1; we may also denote by X, as before, the difference of level $x_{n+1}-x_1$ of the extreme strata; substituting h instead of H_{n+1} to represent the height of the barometer at the upper station, and denoting by H that at the lower station, we shall have

$$X=Mn\frac{\left(\log.\frac{H}{h}+\log.\frac{g_1}{g_{n+1}}\right)}{SC^1}.$$

It is also necessary to find the value of the ratio $\frac{g_1}{g_{n+1}}$ which is that of gravity in the two extreme stations. Since the force of gravity is reciprocally as the square of the distance from the centre of the earth, we shall have $\frac{g_1}{g_{n+1}} = \frac{x^2_{n+1}}{x_1^2}$; but, since X is the difference of level of the two stations, we have $x_{n+1} = x_1 + X$. Consequently, $\log. \frac{g_1}{g_{n+1}} = 2 \log. \left(1 + \frac{X}{x_1}\right)$ x_1 is the distance from the centre of the earth to the lower station; the difference of level X being always extremely small compared with that distance, we may limit ourselves to take for x the mean radius of the terrestrial surface, the value of which in yards is 6962074; and representing it by a, we shall have with sufficient accuracy,

$$\log. \frac{g_1}{g_{n+1}} = 2 \log. \left(1 + \frac{X}{a}\right),$$

consequently,

$$X = \frac{Mn\left\{\log. \frac{H}{h} + 2\log. \left(1 + \frac{X}{a}\right)\right\}}{S C_1}.$$

Before we investigate the value of $S C_1$, we may apply to each of the co-efficients C_1, C_2, C_3, the correction respecting the variation of gravity for different latitudes; this correction, of which the detail has been given at page 180, will consist in multiplying each of them by the factor $1 - 0.002837 \cos. 2\psi$, ψ being the latitude; and as this factor is common to all the co-efficients, since all the points of the column of air are situated in the same vertical line, and may be regarded, therefore, as at the same latitude, it appears that $S C_1$ will become $(1 - 0.002837 \cos. 2\psi) S C_1$, and by transferring this correction into the numerator by developing it into a

series, as we have already done at page 180, we shall have

$$X = \frac{M_{n}(1+0.002837 \cos.2\psi)\left\{\log.\frac{H}{h}+2\log.\left(1+\frac{X}{a}\right)\right\}}{SC_{1}}.$$

The value of SC_1 must now be found; but from the general expression for the co-efficient C, which has been determined at page 186, it is evident that

$$SC_1 = A(1-\overset{\text{yd.}}{0}.0009628).\left(\frac{g_1}{1+0.0022\,t_1}\right.$$
$$\left.+\frac{g_2}{1+0.0022\,t_2}+\frac{g_3}{1+0.0022\,t_3}+\&c.\right)$$

To accomplish this summation in a strict manner, it would be necessary to know the law of the decrease of temperature in the atmosphere. This law is subject to many irregularities. But generally, at small heights, such as those at which barometrical observations are made, it is a very slow arithmetical progression. We shall, therefore, not be far from the truth, by supposing all the temperatures t_1, t_2, t_3, equal to each other, and to the mean temperature between those of the extreme strata; that is, to $\frac{t_1 + t_{n+1}}{2}$. This supposition will increase the temperatures of the upper strata, but it will diminish those of the lower ones, which produces a kind of compensation. By this means the factor dependent upon the temperature becomes common to all the terms of SC_1, and by writing T instead of t_1, and t for t_{n+1}, analogous to the notation which we have adopted for H and h, we shall have

$$SC_1 = \frac{A(1-\overset{\text{yd.}}{0}.0009628}{1+\left(\frac{T+t}{2}\right)0.0022}g_1\left(1+\frac{g_2}{g_1}+\frac{g_3}{g_1}\right.$$

$+\frac{g_4}{g_1}+\&c.$; but gravity being reciprocally as the

square of the distance from the centre of the earth, we shall have $\frac{g_2}{g_1} = \frac{x_1^2}{x_2^2}$, $\frac{g_3}{g_1} = \frac{x_1^2}{x_3^2}$, $\frac{g_4}{g_1} = \frac{x_1^2}{x_4^2}$, &c. and so on in succession. Now, since the difference of the distances of two consecutive strata is D, $\frac{g^2}{g_1} = \frac{x_1^2}{(x_1+D)^2}$, $\frac{g_3}{g_1} = \frac{x_1^2}{(x_1+2D)^2}$, $\frac{g_4}{g_1} = \frac{x_1^2}{(x_1+3D)^2}$, &c. by performing the division algebraically, in each of these terms, they may be reduced to a series, disposed according to the powers of $\frac{D}{x_1}$. We may limit ourselves to the first power of this ratio, which will be sufficient for the object which we have in view, and we shall have

$$\frac{g_2}{g_1} = 1 - \frac{2D}{x_1},\ \frac{g_3}{g_1} = 1 - \frac{4D}{x_1},\ \frac{g_4}{g_1} = 1 - \frac{6D}{x_1},\ \&c.;$$

so that the required sum will become,

$$1 + \frac{g_2}{g_1} = \frac{g_3}{g_1} + \&c. = n - \frac{2}{x_1}\ (D + 2D + 3D + \ldots nD).$$

The part included between the parenthesis constitutes an arithmetic progression, the common difference of which is equal D, and the number of terms n. The sum will therefore be, $\frac{n(n+1)D}{2}$. Now, since D is the thickness of one of the beds, or strata, and n their number, nD is the difference of level of the extreme stations, a difference which has been represented by X; we shall therefore have $1 + \frac{g_2}{g_1} + \frac{g_3}{g_1} + \&c. = n\left\{1 - \left(1 + \frac{1}{n}\right)\frac{X}{x_1}\right\}$.

We see here again the factor n, which remained in the numerator of the expression for the difference of level. By substituting this result in S C_1, it will be sufficient to take instead of x the mean radius of the earth, which has previously been denoted by a. We have

already made use of this simplification; besides, as the number of strata contained in the column is greater as their thickness is less, we ought to neglect the term $\frac{1}{n}$ with respect to those which have not n for a divisor. For, since we finally make D equal to nothing, n will then necessarily be infinite; we shall therefore have

$$S\,C_1 = \frac{A\,(1 - \overset{\text{yd.}}{0.0009628})\,g_1\,n\left(1 - \frac{X}{a}\right)}{1 + \frac{(T+t)}{900}}$$

This transformation of the co-efficient of $T+t$ does not change its value; it only renders it more commodious for calculation. This value of $S\,C_1$ being known, it may be substituted in the general expression for X; in which n disappears by being common to both terms; and there remains

$$X = \frac{M\,(1 + 0.002837\cos.2\psi)\left\{1 + \frac{(T+t)}{900}\right\}\left\{\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right\}}{A\,g_1\,(1 - 0.0009628).\left(1 - \frac{X}{a}\right)}.$$

The factor $1 - \frac{X}{a}$ may be changed into the numerator by division; for $\frac{1}{1 - \frac{X}{a}} = 1 + \frac{X}{a} + \frac{X^2}{a^2} + \&c$; thus, by taking only the first power of $\frac{X}{a}$, which will be sufficient in all cases, we shall have simply

$$X = \frac{M}{A\,g_1(1 - 0.0009628)}\Big[(1 + 0.002837\cos.2\psi)\left\{1 + \frac{(T+t)}{900}\right\}$$
$$\left\{\log.\frac{H}{h} + 2\log.\left(1 + \frac{X}{a}\right)\right\}\left\{1 + \frac{X}{a}\right\}\Big].$$

This equation, containing X in both its members, seems not yet to be completely resolved; but it may be

remarked, that the X in the second member is divided by a, which is always extremely great with respect to X. There is therefore no necessity in calculating these terms, to know X very accurately, but only nearly. Hence the calculation may be divided into two parts. The value of X may be first calculated by neglecting these terms, and then use may be made of that value in calculating them; by uniting these two results the complete value of X will be obtained.

That we may be able to apply the formula which has just been obtained, there only remains the constant co-efficient A to be determined; now, by referring to page 184, where it was first introduced, it will appear, that by calling δ the ratio of the densities of dry air and mercury, under the pressure H and the temperature t, in a place where the latitude is ψ and the gravity g, we have generally $\delta = \frac{A\ (1 - 0.002837 \cos.\ 2\,\psi)\ g\ H}{1 + 0.0020833\,t}$.

The most simple means of finding A is that of weighing very exactly known quantities of air and mercury, under a determinate pressure and temperature, and at a place the latitude and height of which are known. This experiment has been made at Paris with the greatest care, by Arago and myself. We found that, at the temperature of melting ice, and under a pressure of .831136 yd. we had $\delta = \frac{1}{10463}$, from which we obtain

$$A = \frac{1}{10463.\,g(1 - 0.002837 \cos.\ 2\,\psi)\ .831136\ \text{yd.}}$$

ψ being the latitude of Paris; consequently, if the modulus of the logarithmic tables, or 2.30258509, be denoted by M, the co-efficient of the barometrical formula, or $\frac{M}{A\,g_1}$ will become

$$\frac{M}{A\,g_1} = 10463\ (1 - 0.002837 \cos.\ 2\,\psi)\ .831136\ M.\ \frac{g}{g_1}$$

If this value be reduced into numbers, by taking $\psi = 48° 50' 14''$, which is the latitude of the observatory, we have $\frac{M}{A g_1} = 20031\overset{\text{yds.}}{.}27435 . \frac{g}{g_1}$, and consequently,

$$\frac{M}{A g_1 (1 - 0.0009628)} = 20050\overset{\text{yds.}}{.}5654 \frac{g}{g_1}.$$

Let r be the height of the lower station above the level of the sea, $a+r$ will be its distance from the centre of the earth. The height of the place where our experiments relative to the weights of air and mercury were made, may be estimated at 65.6 yards above the level of the sea; its distance from the centre of the earth was therefore $a+65.6$. From this, the ratio of the gravities $\frac{g}{g_1}$ is very nearly equal to $\frac{(a+r)^2}{(a+66)^2}$, an expression which is reduceable to $\left(1 - \frac{131.2}{a}\right) . \left(1 + \frac{2r}{a}\right)$, by developing the two squares and limiting ourselves to the first powers of $\frac{65.6}{a}$, and $\frac{r}{a}$. The first factor $1 - \frac{131.2}{a}$ may be reduced into numbers by taking $a = 6962074$ yards, the number already used. It will diminish the barometrical co-efficient by .3828 yd. which gives $\frac{M}{A g_1} = 20050\overset{\text{yds.}}{.}1826 \left(1 + \frac{2r}{a}\right)$.

This co-efficient may also be determined *à posteriori*, by comparing observations of the barometer with differences of level measured trigonometrically. A great number of very accurate observations, made by M. Ramond in this manner, have given 20052.2496 yds. for the value of the co-efficient, which we have found equal to 20050.1826 yds. from the weights of air and mercury. This agreement proves in a positive manner the accuracy

of the formula, and that of the data upon which it is founded.

There might even be obtained from this agreement, a confirmation of the decrease of gravity in a vertical line. In fact, should not the effect of that decrease be considered, the barometrical observations of M. Ramond would give 20114.5848 yds. for the co-efficient, instead of 20050.1826 yds. as found above from the comparative weights of air and mercury. The difference cannot be attributed to the value which we have assigned to the humidity of the air; for this value is rather too great than too little; and besides, the above difference would not disappear even by supposing the air in a state of extreme humidity, since this supposition, by doubling the correction which has already been made for this object, will only cause 19.2911 yds to be added to 20050.1826, which would be 20069.4737 yds, still much inferior to 20114.5848. It must therefore be acknowledged that the decrease of gravity, though but inconsiderable in the limits in which barometrical observations are made, becomes sensible; and the agreement of the results, when this decrease is taken into the account, demonstrates its reality.

The inequality of temperature in the extreme beds of the column of air which is measured, is communicated to the barometer used, and requires a reduction in the observed heights. In fact, mercury, like other bodies, is condensed by cold and expanded by heat. This variation from the freezing point to 212° of Fahrenheit's thermometer is uniform, according to the experiments of Gay-Lussac, and equal to $\frac{1}{9742}$ for each degree of that thermometer, conformably to the experiments of M. M. Lavoisier and Laplace, which agree with those of the Royal Society of London. Thus, when

the height of the barometer is observed in the colder station, the column of mercury which is condensed ought to appear a little shorter than if it had been observed at the warmer, which is generally the lower. To bring those heights to the same terms, the length of the mercurial column, at the upper station, must be increased in consequence of the different temperatures of the mercury, and proportionally to the condensation which ought to result from it; that is, if the observed length be h', there must be taken, $h=h'\left(1+\frac{T-t}{9742}\right)$.

This supposes that the mercury of the barometer has the same temperature as the surrounding air; but this is not always the case, and the temperatures are sometimes very different. If, at each station where this circumstance is met with, we wished to wait until the barometer had acquired the same temperature as the ambient air, we should be obliged to wait several hours before we should be able to make the observation; for these changes are only completed with great slowness. To avoid this inconvenience, the temperature of the mercury, in the barometer, is measured by means of a small thermometer adapted to the former instrument; and the temperature indicated by this thermometer at the two stations, is that which must be used in the reduction of the barometers to the same temperature. Suppose that it marks (T) degrees at the lower station, (t) at the upper, and that the length of the mercurial column observed at this last station is h', we are to take $h=h'\left(1+\frac{(T)-(t)}{9742}\right)$. By resuming the preceding considerations, the definitive formula for the measure of heights by barometrical observations, according to our experiments, will be

$$X=20050.1826\ (1+0.002837\ \cos.\ 2\,\psi)\ \left[\left(1+\frac{2r}{a}\right)\right.$$

$$\left(1 + \frac{(T+t)}{900}\right) . \left(1 + \frac{X}{a}\right) . \left\{ \log . \frac{H}{h} + 2 \log . \left(1 + \frac{X}{a}\right) \right\} \Big],$$

in which ψ is the latitude of the place, h and t the barometrical height and temperature at the upper station, H and T the analogous quantities at the lower station, r the height of that same station above the level of the sea, expressed in yards, and a the mean radius of the earth, expressed in the same measures; that is, equal 6962074 yds.

By means of the preceding formula, the difference of level of any two places may be very accurately determined from barometrical observations; but these observations must be made with great care, and very good instruments, without which great errors may be committed. Calm weather, and near the middle of the day, should be chosen for the observations as often as possible. One observer should be fixed at each station with instruments previously compared. There each of them observes, at the time appointed, the height of the barometer; he notes at the same instant the state of the thermometer attached to the barometer, in order to have the temperature of the mercury; and also that of a very sensible detached thermometer, exposed in the shade as well as the barometer, and intended to give the temperature of the air. These observations should be repeated every quarter of an hour, the watches having been well regulated by each other, until a certain number of obervations have been attained, ten or twelve, for example. Then the two observers meet and compare again their instruments, in order to ascertain whether they have suffered any accident. If they are found to correspond accurately, the mean of the observations made at each station is to be taken, and the difference of level calculated with these means. If the operation have been

performed with all the precautions above described, the result will only be susceptible of very small errors, arising from accidental irregularities of pressure and temperature of the atmospheric strata: errors which may be made to disappear by their reciprocal compensations, by repeating the observations on different days, and taking an arithmetical mean between all the results. By thus uniting five or six series of corresponding observations, made with good thermometers, and with a barometer mounted with a nonius which gives at least the $\frac{1}{250}$th of an inch, we may answer for two or three yards in the greatest heights.

If, by a long series of observations made at the same place, the mean height of the barometer and the mean temperature of the atmosphere be determined, the formula will give the height of that place above the level of the sea, or any other determinate point. For this, we must also have the mean height of the barometer and thermometer at this second point, and then calculate by the formula as we should do for two stations where we had corresponding observations. This supposes that the mean temperature at the surface of the earth always remains constant, as well as the height of the barometer at each place, and it is possible that these elements may suffer some variations; but the invention of the barometer and thermometer being of too modern a date to determine this, we may, at least, without sensible error, regard them as constant during an interval of some years.

For performing the calculation, the mean height of the barometer at the level of the sea must be known. According to the experiments of Sir George Schuckburg, which are regarded as very accurate, it is 30.035 inches, or .834307 of a yard, at the latitude of 50°, the mean temperature of the air and of the barometer being 55°04

of Fahrenheit's thermometer. At Paris, at the level of the mean waters of the Seine, under the Pont Royal, the mean height of the barometer is .831136 yds., and the mean temperature 53°.6; with these given quantities, and a long series of good observations made at the same place, the height of that place above the level of Paris, or of the sea, might be found.

Barometrical observations, calculated in this manner, and combined with the latitude and longitude, would serve to determine the position of the different points of the earth's surface. In fact, the two co-ordinates, hitherto used, only determine the projection of places upon the surface of the globe; they do not make their elevation known, which the height of the barometer may serve to indicate. For this purpose, there should be made at each place a series of observations with the barometer and thermometer for several years, in order to ascertain the mean temperature and the mean height of the mercury. Good instruments only should be employed, and they should be well compared with each other.

Similar operations, which might easily be extended to the whole of Europe, would give, for this fine portion of the globe, a complete levelling, and much more extended than any obtained from trigonometrical measurements. It would distinctly mark the directions of the chains of mountains, the declivities of rivers, and, above all, it would show the form of the earth's surface much better than simple descriptions. Physical geography, too little cultivated at present, would undoubtedly derive great advantage from it.

To induce observers to undertake this work, I have subjoined a table, which, without any other calculation than a single subtraction of two numbers, will give the elevation of places, and the difference of level, from the observed heights of the barometer and thermometer.

This Table is calculated from a modification of the formula which has already been indicated, and which consists in uniting the correction relative to the decrease of gravity with the constant co-efficient of the formula, which raises it to 20114.5848 instead of 20050.1826, as shall now be shown.

For this purpose, the rigorous formula may be resumed, making, for the sake of abridging the work, $N = 20050.1826\left(1+\frac{2r}{a}\right)(1+0.002837 \cos. 2\psi)$ $\left(1+\frac{(T+t)}{900}\right)$, and we shall have

$$X = N\left\{\log.\frac{H}{h} + 2\log.\left(1+\frac{X}{a}\right)\right\}.\left(1+\frac{X}{a}\right).$$

As X is contained in both members of this equation, it must be eliminated. To effect this, let the logarithm of $1.+\frac{X}{a}$ be developed, and the first power only be taken, whence we have $\frac{X}{Ma}$, M being the modulus of the log-tables, or 2.3025850; then, by performing the multiplication with the factor $1.+\frac{X}{a}$, and always limiting ourselves to the first power, we shall have

$$X = N.\log.\frac{H}{h} + \frac{N}{a}\left(\log.\frac{H}{h}+\frac{2}{M}\right)X,$$

from which we readily obtain

$$X = \frac{N\log.\frac{H}{h}}{1-\frac{N}{a}\left(\log.\frac{H}{h}+\frac{2}{M}\right)}.$$

The denominator of the second member is nearly equal to unity; for the co-efficient $\frac{N}{a}$, by which the se-

cond term is multiplied, is a very small fraction, and very little different from $\frac{1}{346}$; and the other factor, log. $\frac{H}{h}+\frac{2}{M}$, can never exceed unity within the limits of barometrical observations; in fact, the quantity $\frac{2}{M}$ is constant and equal to .8685830; the other term, log. $\frac{H}{h}$ which is variable, is much smaller still, since even by supposing $H=.831136$ yd, and $h=.65616$ yd, which answers to a difference of level of nearly 2190 yds, its numerical value is only $= 0.1026623$. We might, therefore, neglect this term, because of its small influence; but it will be better to retain it, and assign to it the mean value obtained from the preceding calculation; for the error which may result from it when h is less than .65616 yd, will always be small in the greatest heights to which we can attain; and that which may take place when h is greater, will be attenuated by the smallness of the log. $\frac{H}{h}$ in the numerator. By this means, the second term of the denominator becomes constant; for, on account of its smallness, we may calculate it with the constant part of N, and then its value, $\left(\text{by using } \frac{N}{a} = \frac{1}{346}\right)$ is 0.0028071. The denominator is, therefore, 1—0.0028071, and, by carrying it into the numerator by division, as a factor, it becomes

$$X = N(1+0.0028071)\log.\frac{H}{h}.$$

In the value of N, a mean value might in the same manner be assigned to r, which expresses the height of the lower station above the level of the sea; for the cor-

rection which results from it, having the radius of the earth for a divisor, is so small, that we might almost reject it altogether, especially in small differences of level; but for the same reason, it will be better to retain it, and assign to it a mean value which approaches to those in which its influence may become sensible. For this purpose, let it be supposed that $r = 1312.32$ yds, which is, perhaps, nearly the mean height at which travellers may have the most frequent occasions to make observations upon the mountains in our climates. We shall thus have $\frac{2r}{a} = \frac{2624.64}{6962074} = 0.00037699$. The constant part of the co-efficient, which was at first 20050.1826, will, therefore, by these transformations, become 20050.1826 $(1+0.00037699).(1+0.0028071)=20114.5848$, and consequently, the expression for the difference of level will become $X=20114.5848$ yds. $(1+0.002837 \cos. 2\psi)$. $\left(1+\frac{(T+t)}{900}\right) \log. \frac{H}{h}$.

This formula possesses all the accuracy which we can hope to attain by barometrical observations. Compared with the rigorous expression for X, it will give only a little more than four yards of error in the height of Chimbaraço, which is about 6430 yards, according to the observations of M. Humboldt; much more will it be sufficient in all the barometrical levellings in which travellers may be interested. It is this simplified formula that has been reduced into a table.

Explanation of the Barometrical Tables.

It may first be remarked, that the factor $1+0.002837$ $\cos. 2\psi$, which depends upon the latitude, will always be extremely small, for it is nothing at 45° of latitude; and at

the equator, or at the pole, where it attains its *maximum*, its second term is also less than $\frac{3}{1000}$th, so that the correction which results from it, will not be the $\frac{3}{1000}$th of the observed height. It might, therefore, be neglected in most observations; but, in order that it may be taken into the account, I have formed a small table of its values for every 5° of latitude. This table shows at once what is to be added to or taken from the difference of level calculated with the other terms of the formula, in order to have regard to this correction. Thus, it appears, for example, that at 45° of latitude, the height does not require any correction; at 40° there must be added to the height calculated, $\frac{1}{2036}$th of its value; at 35° it is $\frac{1}{1036}$, and so on; on the contrary, from 45° to the pole, the fraction of the height contained in the table must be subtracted. An application of this correction will be made by and by to some numerical examples.

It therefore only remains to consider the other terms of the expression for X, which becomes

$$X = 20114\overset{\text{yd.}}{.}5848\left(1 + \frac{(T+t)}{900}\right)\log.\frac{H}{h}.$$

The difficulty of reducing this expression into a table, arises from the circumstance of its containing three variable elements, $T+t$, H, h; that is, the sum of the detached thermometers, and each of the observed heights of the barometer at the two stations, after being corrected for the expansion of the mercury. But this difficulty may be eluded by a simple artifice, which may be useful in many other cases; this artifice consists in decomposing the log. $\frac{H}{h}$ into two terms of the same form, *viz.* log. $\frac{.831136\,\text{yd.}}{h}$ — log. $\frac{.831136\,\text{yd.}}{H}$. It is evident, in fact,

that the difference of these two terms is equal to log. $\frac{H}{h}$;
but these terms being both of the same form, may be given by the same table. By introducing them into the value of X, we shall have

$$X = 20114\overset{yds}{.}5848\left(1+\frac{(T+t)}{900}\right).\left(\log.\frac{.831136\text{ yd.}}{h}\right.$$
$$\left.-\log.\frac{.831136\text{ yd}}{H}\right)$$

It appears that it will be sufficient to construct a table of the quantity,

$$20114\overset{yds.}{.}5848\left(1+\frac{(T+t)}{900}\right)\log.\frac{.831136\text{ yd}}{h},$$

in which we may give to $T+t$ and h, all the values which may be met with in common observations. Then, when the values $T+t$, h and H, are given for a particular observation, we may enter the table first with $T+t$ and h, and we shall find a number; then with $T+t$ and H we shall find another number. The difference of these two numbers will be strictly the difference of level X. The table which is subjoined to the end of this chapter is thus constructed. The first vertical column of each page contains the heights of the barometer for every thousandth part of a yard, from .831 yd,* (29.916 inches) to .650 yd, (23.4 inches,) which answers to about 2310 yards in the difference of level. These values are supposed to be brought to the same temperature, for example, to that of the lower station; so that if the lengths of the observed columns of mercury are H and h, and their

* As the first column of the Table contains only three figures, the last three of the number .831136 are necessarily omitted; and for the value of these, *viz.* 136, see the note opposite the first page of the Table. *Translator.*

temperatures (T) and (t), the table must be entered with the numbers $T+t$, H and $h\left(1+\frac{(T)-(t)}{9742}\right)$.

It would be almost as simple to reduce the temperatures of the two columns of mercury to that of melting ice, which would render the calculations uniform.

The first horizontal division of the table, entitled SUM OF THE DETACHED THERMOMETERS, presents the values of $T+t$, calculated for every degree of Fahrenheit's thermometer, from 20° to 74°.

Though the extent of this table is confined to the limits which have been assigned, its use may be extended to all possible cases, by means of a very simple artifice, which shall be subsequently explained by applying it to the calculation of the height of Chimboraço. For the present, we shall consider the common case, in which the table is to be entered with such values of $T+t$, H and h, as are comprised in it.

Having the height of the barometer at the upper station, the small correction is to be made for the expansion of mercury. and we shall then have h; we must then find in the first column of the table the number which approaches nearest to it, and follow the horizontal line corresponding with this number, until we arrive at the column which answers to $T+t$; the number which is found at the meeting of these two columns will be the first term of the formula expressed in yards.

The same operation is to be repeated with the value of H relative to the lower station, always employing the same value of $T+t$; and the second term of the formula will be obtained in yards.

If H be less than .831 yd. the second term is to be subtracted from the first; and the difference will be the value of X, or the difference of level required.

But if H be greater than .831 yd. the second term must be added to the first.

Let it be supposed, for example, that the following quantities were given.

	Height of the Barometer.	Detached Thermometer	Attached Thermometer	Latitude.
Lower Station	0.8202 yd.	$+64^\circ.4$	$+64^\circ.4$	50°
Upper Station	0.65495	$+46^\circ.4$	$+46^\circ.4$	

The difference of the temperatures of the mercury is 18°; the correction of the upper barometer is therefore $\frac{.65495 \times 18}{9742} = .00121$ yd. additive; we have therefore

$T+t=46^\circ.4$ $H=0.8202$ yd. $h=0.65616$ yd.

With $T+t=46^\circ.4$ and $h=0.65616$ the table gives 2169.97

With $T+t=46^\circ.4$ and $H=0.8202$........... — 120.16

Difference 2049.81

Add 1.5*

2051.31

Correction for the latitude $-\frac{1}{2030}$ .... — 1.01

The required difference of level in yards 2050.30

This table also furnishes the means of ascertaining the heights of places above the level of the sea, when, from a long series of observations, the mean heights of the barometer and thermometer are known. It will be sufficient to combine these given quantities with those analogous to them at the surface of the sea. Now, according to the observations of M. Shuckburg, which are reckoned very exact, the mean height of the barometer at the level of the ocean, for the latitude of 50°, is .83431 yd.; and the mean temperature, at the same latitude, $55^\circ.04$ of Fahrenheit's thermometer.

* See the note opposite the first page of the Table. *Translator.*

Let these values be compared with their corresponding ones at Geneva, latitude 46°.12′. According to the observations of the celebrated Saussure, the temperature at Geneva is equal to 53°.6 of Fahrenheit. The mean height of the barometer in that city, according to M. Cotte, is .79461 yd. This result is derived from a series of observations continued for 14 years.

The temperatures of the mercurial columns are here the same as that of the air; their difference is 55°.04 − 53°.6 = 1°.44; consequently

$$h = .79461 + \frac{.79461 \times 1°.44}{9742} = .79473 \text{ yd.}$$

With $h = .79473$ yd. and $T+t = 44°.64$,
the table gives* .... 409.17 yds.
With $H = .83431$ yd. and $T+t = 44°.64$.. + 36.64
H being greater than .831, the sum is to be
taken .... = 445.81
Correction for the mean latitude $\frac{1}{2030}$ .. − .22
Height of Geneva above the level of the sea = 445.59 yds.

* As the table is only calculated for every degree of the thermometer, and every thousandth part of a yard for the height of the barometer, when smaller fractions than these are to be used, their values must be obtained by means of proportional parts. For example, the value of h at Geneva being .79473 yds. the corresponding number is comprised between those answering to .795 and .794; look in each of these lines for the numbers which correspond to $T+t = 44°$; in the first, it is 405.77 yds. with a difference of .432 for 1°; this will give 0.28 nearly, for 0.64: thus, the number in this line which answers to 44°.64 is 406.05 yds. The same for the following line, from which the analogous number is 417.31 yds. with a difference of .444 for 1°, which gives nearly .29 for 0°.64: thus the number in this line corresponding to 44°.64 is 417.6 yds. Subtracting the less of these numbers from the greater, the difference 11.55 is the variation in altitude, for a change of one thousandth part of a yard in the

These two examples are sufficient for the cases where H, h and $T+t$ are comprised within the limits of the table, as the calculation will always be the same. We shall pass to the case when one of these quantities is without the limits of the table; let it be $T+t$.

It very seldom happens in observations that the sum of the detached thermometers is less than 20° above the freezing point of Fahrenheit's scale, or more than 74°: but when this does take place, the operation is as follows:—

If $T+t$ be less than 20°, let the number of degrees necessary to make it equal to that be added. Suppose t' to be this number. With the observed barometrical columns H, h and $T+t+t'=20°$, enter the table as usual; but when the partial heights have been found in yards, subtract from each of them the product of t' multiplied by the difference for 1°, which is found in the same horizontal line. Thus you shall obtain the same numbers as the table would have given if it had extended below 20°.

An artifice analogous to this must be employed, if the sum of the detached thermometers, the degrees of each estimated from the freezing point, exceeds 74° of Fahrenheit. In this case, the number of degrees necessary to bring it to 74° is to be subtracted; and to each of the partial results

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height of the barometer at this temperature: but from .79472 to .795 the change is 0.00028 yds.; this gives 3.234 yds., which must be added to 406.05, answering to .795 yd.; we shall then have 409.284 yds: this increased by 1.5 (see the note opposite the first page of the Table) gives 410.784 yds. the required number. These reductions are taken at sight from the table, and, with a little experience, they are much easier to execute than to explain. It is in this manner, that in the use of logarithms, we find the logarithm of a number which is not in the tables, but contained between two numbers hat are comprised in them.

obtained for H and h, the product of this excess by the difference for 1° is to be added.

The reasons upon which these methods of operation are founded is, that the numbers in the same horizontal line of the table increase by the same quantity for each degree. The common difference of this arithmetical progression is expressed in the last column, entitled, *Differences for* 1°.

Nevertheless, as has already been remarked, we shall very seldom have occasion to make use of these reductions.

It will not be the same for those which relate to H and h. These quantities may often be out of the limits of the table, but they may always be brought within them by an artifice so simple, that it will be better to make use of it than to increase the extent of the table.

First, if H exceed .831 yd. which only happens very seldom, the difference will always be very small; for the greatest heights of the barometer that are observed at the surface of the earth, do not exceed .853 yd.; in this case, the two observed heights H and h, may be diminished in the same proportion; that is to say, we may subtract from each of them $\frac{1}{100}$ of its value, or $\frac{1}{10}$ if it be necessary. Then H will be found in the table, and the operations may be performed with these transformed values the same as in the common case.

The reason for this proceeding is that the formula $X = \overset{\text{yds.}}{20114.5848} \left(1 + \frac{(T+t)}{900}\right) \log. \frac{H}{h}$, contains only the ratio $\frac{H}{h}$ of the two barometrical columns, a ratio which will not be altered by either increasing or diminishing both terms in the same proportion. If it be not sufficient to subtract $\frac{1}{100}$th from H to bring it within the table, $\frac{1}{10}$th

may be subtracted, and then it will certainly be within the limits. Suppose that we had, for example,

$$H=0.85 \text{ yd. and } h=0.76 \text{ yd.}$$

Subtracting $\frac{1}{10}$th............0.085 .... 0.076

The corrected values of $H=0.765$......$h=0.684$

which are both found in the table. With these values and that of $T+t$, look for the partial heights as usual, and their difference will give the difference of level.

Any other fraction might be equally subtracted. Resuming this example,

$$H=0.85 \text{ yd.} \ldots\ldots h=0.76 \text{ yd.}$$

Subtracting $\frac{1}{100}$th ....0.0085 ....0.0076

$$H=0.8415 \ldots\ldots h=0.7524$$

As H does not yet enter into the table, subtract again $\frac{1}{100}$th 0.008415....... 0.007524

The corrected values are $H=0.833085$ and $h=0.744876$

These values will give the same difference of level as the two first which were obtained by subtracting $\frac{1}{10}$th; this may easily be proved from the table, by calculating with each of them separately.

Let us now examine the case in which h is less than .65 yd. the superior limit of the table. In this case it might easily be brought within the limits by an analogous operation, that is, by multiplying the two terms of the fraction $\frac{H}{h}$ by the same number; but this operation may have the inconvenience of causing H to exceed the table, by making it greater than .835 yd. In order to avoid this inconvenience, the operation may be performed in the following manner. Resuming the formula

$$X=20114.5848^{\text{yds.}}\left(1+\frac{(T+t)}{900}\right).\left(\log\frac{.831^{\text{yd.}}}{h}-\log.\right.$$

$\left.\frac{.831^{yd.}}{H}\right)$; make Log. $\frac{.831^{yd.}}{h} = \log. \frac{.831\,(1+\frac{1}{4})}{h\,(1+\frac{1}{4})} = \log.$ $\frac{.831^{yd.}}{h(1+\frac{1}{4})} + \log.\frac{5}{4} = \log.\frac{.831^{yd.}}{h(1+\frac{1}{4})} + \log.\frac{.821^{yd.}}{.66491}$; we shall then have $X = 20114.\overset{yds.}{5848}\left(1 + \frac{(T+t)}{900}\right).\left(\log.\right.$ $\left.\frac{.831^{yd.}}{h(1+\frac{1}{4})} - \log.\frac{.831^{yd.}}{H} + \log.\frac{.831^{yd.}}{66491}\right).$

The three terms which constitute the value of X may be taken in the table. Let $h = .52^{yd.}$, we shall have $h(1+\frac{1}{4}) = .65^{yd.}$, and h will be brought within the limits of the table. This proceeding will therefore suffice when the barometrical column at the upper station is not less than $.52^{yd.}$, which answers to an altitude of about 4155 yards above the level of the sea. This case requires only the addition of one term more, and it very seldom happens that we are elevated to greater heights, at least in Europe.

EXAMPLE. M. Humboldt made the following observations upon the mountain of Quindiù, in the kingdom of New Grenada, at the point of separation of the waters which flow on the one side into the Atlantic ocean, and on the other into the Pacific Ocean.

	Height of the Barometer.	*Detached Thermometer.*	*Attached Thermometer.*	*Latitude.*
Upper Station	yd. 0.557537	65°.75	68°	5°.
At the level of the Pacific Ocean, at the same time	0.834356	77.54	77.54	

Here we have $h = 0.557537 \left(1 + \frac{9^{\circ}.54}{9742}\right) = \overset{yd.}{0.558083}$

Add $\frac{1}{4}$th of h 0.139521

Value of h brought within the table, ... $h = 0.697604$

With $h = 0.697604$*	....	the table gives	1663.65
$H = 0{\cdot}834356$	& $T + t = 79^{\circ}.29'$	....	+ 38.20
The const. 0.66491	....	....	2119.73
	Add	....	1.5 †
	Sum	....	3823.08

Correction for latitude $+ \frac{1}{358}$ .... + 10.67

Height above the ocean .... 3833.75 yds

The same artifice would also serve for points still more elevated, for if h was not within the limits of the table when we had multiplied it by $\frac{5}{4}$, nothing prevents us from multiplying again by $\frac{5}{4}$, provided that, instead of the term $\log. \frac{.831}{.66491}$yd. we take its double. In effect, we have evidently $\log. \frac{.831}{h} = \log. \frac{.831\,(1+\frac{1}{4})^2}{h\,(1+\frac{1}{4})^2} = \log$ $\frac{.831}{h\,(1+\frac{1}{4})^2} + 2 \log. \frac{5}{4} = \log. \frac{.831}{h\,(1+\frac{1}{4})^2} + 2 \log. \frac{.831}{.66491}$, then we shall have $X = \overset{yds.}{20114.5848} \left(1 + \frac{(T+t)}{900}\right)$.

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* The reduction of the numbers from French to English measures increases the number of decimal places, and renders the calculations much more troublesome than those which result from actual observations, in which the observed number seldom exceeds three places of decimals. It may also be remarked, that, in reducing the height of the barometer as observed at one station to what it would have been at the temperature of the other, it will generally be a sufficient approximation to divide by 9600 instead of 9742, and to limit ourselves to four or five places of decimals, which will facilitate the calculations. *Translator*.

† See the note opposite the first page of the Table.

$$\left(\log.\frac{.831}{h\,(1+\frac{1}{4})^2} - \log.\ \frac{.831}{\mathrm{H}} + 2\log.\frac{.831}{.66491} \right).$$

This formula is not more difficult to calculate than the preceding one. It will be sufficient for any altitude equal to that of Chimboraço; but, if we wish to exceed all the heights accessible to man, even that of the ascension of Gay-Lussac, we shall have only to take

$$\mathrm{X} = 20114\overset{\text{yds.}}{.}5848\left(1 + \frac{(\mathrm{T}+t)}{900}\right).\ \left(\log.\ \frac{.831}{h\,(1+\frac{1}{4})^3} - \log.\ \frac{.831}{\mathrm{H}} + 3\log.\frac{.831}{.66491} \right);$$

a formula which will also be very easy to calculate. The following is an example of it applied to the measure of the height of Chimboraço, by M. de Humboldt.

	Height of the Barometer.	*Detached Thermometer*	*Attached Thermometer.*	*Latitude.*
Upper Station	.412588 yd	29°.12	50°	
At the level of the Pacific Ocean	.833323 yd	77°.54	77°.54	1° 45′

Here we have

$$h = .412588\left(1 + \frac{27^\circ.54}{9742}\right) = .412588 + .002827 = .415415$$

To this add $\frac{1}{4}$th of h .... =	.103854
	.519269
The result not being comprised in the table, add $\frac{1}{4}$th of its value again	.129817
	.649086
This result is very near the limits of the table, let $\frac{1}{4}$th of its value be added again......	.1622725
We have finally .... h =	.8113585

With $h = .811358$	& $T+t=42°.66$	the table gives	218.566
$H = .833323$		......	+ 25.548
Const. quan. = .66491		......	2040.174
			2040.174
			2040.174
	Add 1.5×3		4.5
	Sum		6369.136
	Correction for latitude $\frac{1}{352}$ ..		+ 18.094
Height of Chimboraço above the sea			6387.23 yds

Lastly, there only remains the case to be examined, in which the two values of H and h are both less than .65 yd.; this is very easy. But the quantities are to be multiplied by the same number, so that the lower station may be found in the table, after which the operation is the same as above.

Example. Suppose that travellers had passed the night at 2624.64 yards of elevation, and that they begin to ascend from this point. At the time of departure the barometer stood at .634 yd. and the heights observed in ascending were all less than that number. Required the calculation by the table.

In order to fix the ideas, suppose we had	$H = 0.634$ yd.	$h = 0.514$ yd.
Adding $\frac{1}{10}$th to each height....	0.0634	0.0514
	$H = 0.6977$ yd.	$h = 0.5654$ yd.

Now H being contained in the table, the calculation is to be performed as in the preceding examples.

Specimens of these different examples have been subjoined to the Table, in order to bring the numerical applications into one view, and avoid the necessity of recurring to the text.

Some general indications, which M. Ramond has deduced from numerous experiments, shall now be speci-

fied; they will serve to show observers what degree of precision they may expect from barometrical measures, according to the state of the atmosphere.

I. The estimated altitudes are generally too little,

When the observation is made in the morning or evening.

When the lower barometer is in a plain, and the upper barometer in a narrow and deep valley;

When the wind blows strongly from the south;

When the weather is stormy; and in this case we may commit great errors.

II. The estimated altitudes, on the contrary, are too great,

When the observations are made between twelve and two or three o'clock, especially during summer, and under an ardent sun.

When the upper barometer is at the top of a mountain, and the lower one in a narrow and commanded pass.

When the north wind blows strongly, particularly if we are on a mountain, and the wind strike against its steep side.

Lastly, an important remark may be added.

Observation proves, that all other circumstances being the same, the mercury is always more elevated in a syphon barometer than in a cistern barometer. M. Laplace has shown that this inequality is an effect of capillary action, which depresses the column of mercury in the cistern barometer, while it is compensated in the two legs of the syphon barometer.

The following table has been calculated by M. Laplace to correct this effect, and the reader will find, at the end of the Barometrical Tables, the Table relative to the correction for latitude.

Table of Depression of the Mercury in the Barometer, arising from Capillary Action.

Interior diameter of the Tubes, in decim. of an Inch.	*Depression in decim. of an inch.*
.07874	.17952
.1181	.11426
.1575	.08027
.1968	.05927
.2362	.04520
.2756	.03470
.3150	.02697
.3543	.02104
.3937	.01654
.4331	.01380
.4724	.01024
.5118	.00806
.5512	.00629
.5590	.00490
.6299	.00382
.6693	.00297
.7087	.00232
.7480	.00170
.7874	.00139

Numerical Examples of the Calculation of Altitudes by the Formula.

$$X = 20114\overset{\text{yds.}}{.}5848\,(1 + 0.002837 \cos . 2\psi) . \left(1 + \frac{T+t}{900}\right) . \left(\log . \frac{.831}{h} - \log . \frac{.831}{H}\right),$$

H = the height of the column of mercury at the lower station, expressed in decimal fractions of a yard.

h = the height of the mercurial column in the upper station, corrected for the expansion of mercury.

$T+t$ = the sum of the temperatures of the air at the two stations, expressed in degrees of Fahrenheit's thermometer.

Case I. When h and H are both comprised in the Table, consult it immediately. Example.

	Height of the Barometer.	*Detached Thermometer.*	*Attached Thermometer.*	*Latitude.*
Lower Station	0.8202 yd.	+ 64°.4	+ 64°.4	50°
Upper Station	0.65495yd	+ 46°.4	+ 46°.4	

$T+t=46°.4$, $H=.8202$ yd, $h=.65495$ yd. $\left(1+\frac{18}{9742}\right)$
$=.65616$ yd.

With $h=.65616$ } & $T+t=46°.4$ { the table gives 2169.97yds
$H=.8202$ } { ....—120.16

Difference 2049.81
Add............ 1.5
2051.31

Correction for Latitude $\frac{1}{2030}$........ — 1.01

Difference of level 2050.30yds

The term H is *subtractive* when its value is less than .831 yd. and *additive* when it is greater than .831 yd.

Case II. When h is not comprised in the table, but exceeds .525 yd. Example.

	Height of the Barometer.	*Detached Thermometer.*	*Attached Thermometer.*	*Latitude.*
Upper Station	.557537yd	+ 65°.75	+ 68°	5°
Lower Station	.834356yd	+ 77°.54	— 77°.54	

$T + t = 79°.29$, $H = .834356^{yd.}$ $h = .557537^{yd.}$

$\left(1 + \frac{9°.54}{9742}\right)$ .... $=0.558083$yds.

Adding $\frac{1}{4}$th of h.......... 0.139521

Value comprised in the table .... $h = 0.697604$

With $h = .697604$		the table gives	1663.65yds.
$H = .834356$	& $T+t=79°.29$	....	+ 38.20
Constant .66491		....	2119.73
	Add...............		1.5
			3823.08
	Correction for Latitude	....	+10.67
	Difference of level	....	3833.75yds.

Case III. When h is less than .525 yd. and greater than .42 yd. Example.

	Height of the Barometer.	*Detached Thermometer.*	*Attached Thermometer.*	*Latitude.*
Upper Station	.429719	29°.12	50°	1° 45′
Lower Station	.833323	77°.54	77°.54	

$T + t = 42°.66$, $H = .833323$, $h = .429719$

$\left(1 + \frac{27°.54}{9742}\right)$ .... $=.432546$

Add $\frac{1}{4}$th of h .... .108136

.540682

Add again $\frac{1}{4}$th of this value .... .135171

$h = .675853$

With $h = .675853$		the table gives	1890.818
$H = .833323$	& $T+t=42°.66$	......	+ 25.548
Const. q. $= .66491$		......	2040.174
Add this term again	....		2040.174
Add 1.5×2	....		3
	Sum	....	5999.714
	Correction for latitude $\frac{1}{352}$	..	+17.045
	Difference of level	......	6016.759yds.

Case IV. When H is not comprised in the table, it may be made to enter it by multiplying or dividing both H and h by the same number; for example, by either adding or subtracting from each $\frac{1}{10}$ of its value.

Let........	$H = .6278$ yd.	$h = .527$ yd.
Add $\frac{1}{10}$......	.06278	.0527
	$H = .69058$	$h = .5797$

Then the altitude is to be calculated with these numbers as in the preceding examples.

If, on the contrary,	$H = .853$ yd.	$h = .795$
Subtract $\frac{1}{10}$	.0853	.0795
	$H = .7677$	$h = .7155$

Then the remainder of the calculation is to be performed with these heights.

If the sum of the detached thermometers is not comprised in the table, the proportional parts contained at the end of each line are to be used.

When the difference of level is very small, the inequality of the temperatures of the two columns of mercury may prevent the real difference of their lengths from being ascertained; and then it will not be known which is to be taken for h and H. But, in this case, it is only necessary to reduce either of them to the temperature of the other; and then the shorter will be h, and the longer H.

Lastly. When an error of a few yards is of no consequence, as in the courses of Botany, it will be sufficient to take for h, H and $T+t$, the even thousandths of a yard and whole degrees only, neglecting the fractions; then the difference of level will not require fifteen seconds of time for the calculation.

OBSERVATION.—As the scale of the *centigrade* thermometer, used by the French, commences at the freezing point, the *zero*, or 0, on that scale, corresponds to the 32nd degree of Fahrenheit's thermometer; and, as $T+t$, in the preceding formulæ and examples, indicate the sum of the two temperatures taken *above* the freezing point, it was necessary, in reducing the degrees from the centigrade to Fahrenheit's scale, to add 32° to each number; consequently, 64° must be subtracted from the sum of the temperatures, as given by Fahrenheit's thermometer, and the Table entered with the remainder, as stated at the head of each page. For instance, in the example relative to the height of Chimboraço, page 214 or 219, the two temperatures as marked by the detached thermometer, according to Fahrenheit's scale, are 29°.12 and 77°.54; and therefore $T+t = (29°.12+77°.54) - 64° = 42°.66$; the number with which the Table is to be entered. But, should the remainder not be found in the Table, the process to be followed is explained at page 209, and applied to the subsequent examples.

The same observation is also applicable to all the other cases.

Translator.

TABLES

FOR

REDUCING ANGLES

From one Plane to Another.

TABLE I.

To Reduce an Angle Observed in an Inclined Plane to the corresponding Horizontal Angle, take a Number in this Table, with the Argument (H + h), *and then a Second Number, with the Argument* H — h).

To Reduce a Horizontal or Spherical Angle to the Angle of the Chords, take in the same Table a first Number, with the Argument (P+Q) *and a Second with the Argument* (P—Q).

ARGUMENTS (H ± *h*), (P ± Q).

M.	0°	1°	2°	3°.	4°.
1	0,000	0,787	3,097	6,928	12,281
2	0,001	0,813	3,148	7,005	12,383
3	0,002	0,839	3,200	7,082	12,486
4	0,003	0,866	3,252	7,160	12,589
5	0,005	0,893	3,305	7,237	12,692
6	0,007	0,921	3,358	7,316	12,796
7	0,010	0,949	3,411	7,395	12,900
8	0,013	0,978	3,465	7,475	13,005
9	0,017	1,007	3,520	7,554	13,110
10	0,021	1,036	3,575	7,634	13,215
11	0,026	1,066	3,630	7,715	13,321
12	0,030	1,096	3,685	7,796	13,428
13	0,036	1,127	3,741	7,877	13,534
14	0,041	1,158	3,798	7,959	13,641
15	0,047	1,190	3,855	8,041	13,749
16	0,054	1,222	3,912	8,124	13,857
17	0,061	1,254	3,970	8,207	13,965
18	0,068	1,287	4,028	8,291	14,074
19	0,076	1,320	4,086	8,375	14,184
20	0,084	1,354	4,145	8,459	14,293
21	0,093	1,388	4,205	8,544	14,403
22	0,102	1,422	4,265	8,629	14,514
23	0,112	1,457	4,325	8,715	14,625
24	0,122	1,492	4,386	8,801	14,736
25	0,132	1,528	4,447	8,887	14,848

M.	0°.	1°.	2°.	3°.	4°.
26	0,143	1,564	4,508	8,974	14,960
27	0,154	1,601	4,570	9,061	15,073
28	0,166	1,638	4,633	9,149	15,186
29	0,178	1,675	4,695	9,237	15,300
30	0,190	1,713	4,759	9,326	15,413
31	0,203	1,751	4,822	9,415	15,528
32	0,217	1,790	4,886	9,504	15,642
33	0,230	1,829	4,951	9,594	15,757
34	0,244	1,869	5,016	9,684	15,873
35	0,259	1,909	5,081	9,775	15,989
36	0,274	1,949	5,147	9,866	16,105
37	0,289	1,990	5,213	9,958	16,222
38	0,305	2,031	5,280	10,050	16,340
39	0,321	2,073	5,347	10,142	16,457
40	0,338	2,115	5,414	10,235	16,575
41	0,335	2,158	5,482	10,328	16,694
42	0,373	2,200	5,550	10,422	16,813
43	0,391	2,244	5,619	10,516	16,932
44	0,409	2,288	5,688	10,610	17,052
45	0,428	2,332	5,758	10,705	17,172
46	0,446	2,376	5,828	10,800	17,293
47	0,467	2,421	5,899	10,896	17,414
48	0,487	2,467	5,970	10,992	17,535
49	0,508	2,513	6,041	11,089	17,657
50	0,529	2,559	6,112	11,186	17,780
51	0,550	2,606	6,184	11,283	17,903
52	0,572	2,653	6,257	11,381	18,026
53	0,594	2,701	6,330	11,480	18,150
54	0,617	2,749	6,403	11,578	18,274
55	0,640	2,797	6,477	11,677	18,398
56	0,663	2,846	6,551	11,777	18,523
57	0,687	2,895	6,626	11,877	18,648
58	0,711	2,945	6,701	11,977	18,774
59	0,736	2,995	6,776	12,078	18,900
60	0,761	3,046	6,852	12,179	19,026

TABLE II.

In the Reduction to the Horizon, the Tangent Number is positive, and the Co-tangent Number is negative. The contrary takes place for the Angle of the Chords.

Angl. D. M.	Tang.	Cotang.		Angl. D. M.	Tang.	Cotang.	
	″	″			″	″	
0 0	0,00	∞	180 0	4 0	0,72	590,68	176 0
10	0,03	14181, 5	50	10	0,75	567,02	50
20	0,06	7090, 8	40	20	0,78	545,19	40
30	0,09	4727, 2	30	30	0,81	524,98	30
40	0,12	3545, 4	20	40	0,84	506,21	20
50	0,15	2836, 3	10	50	0,87	488,74	10
1 0	0,18	2363, 8	179 0	5 0	0,90	472,44	175 0
10	0,21	2025, 9	50	10	0,93	457,57	50
20	0,24	1772, 6	40	20	0,96	442,86	40
30	0,27	1575, 7	30	30	0,99	429,42	30
40	0,30	1418, 1	20	40	1,02	416,77	20
50	0,33	1289, 1	10	50	1,05	404,84	10
2 0	0,36	1181, 7	178 0	6 0	1,08	393,58	174 0
10	0,39	1090,80	50	10	1,11	382,92	50
20	0,42	1012,80	40	20	1,14	372,83	40
30	0,45	945,30	30	30	1,17	363,24	30
40	0,48	886,20	20	40	1,20	354,14	20
50	0,51	834,05	10	50	1,23	345,49	10
3 0	0,54	787,70	177 0	7 0	1,26	337,24	173 0
10	0,57	746,22	50	10	1,29	329,38	50
20	0,60	708,89	40	20	1,32	321,87	40
30	0,63	675,11	30	30	1,35	314,70	30
40	0,66	544,40	20	40	1,38	307,84	20
50	0,69	616,37	10	50	1,41	301,27	10
4 0	0,72	590,68	176 0	8 0	1,44	294,98	172 0
	Cot.	Tang.	D. M. Angle.		Cot.	Tang.	D. M. Angle.

Angl. D. M.	Tang.	Cotang.		Angl. D. M.	Tang.	Cotang.	
	″	″			″	″	
8 0	1,44	294,98	172 0	14 0	2,53	167,99	166 0
10	1,47	288,93	50	10	2,56	165,99	50
20	1,50	283,14	40	20	2,60	164,04	40
30	1,53	277,56	30	30	2,63	162,14	30
40	1,56	272,20	20	40	2,66	160,27	20
50	1,59	267,05	10	50	2,69	158,45	10
9 0	1,62	262,09	171 0	15 0	2,72	156,68	165 0
10	1,65	257,30	50	10	2,75	154,93	50
20	1,68	252,68	40	20	2,78	153,23	40
30	1,71	248,23	30	30	2,81	151,56	30
40	1,74	243,93	20	40	2,84	149,93	20
50	1,77	239,78	10	50	2,87	148,33	10
10 0	1,80	235,77	170 0	16 0	2,90	146,77	164 0
10	1,83	231,88	50	10	2,93	145,23	50
20	1,86	228,12	40	20	2,96	143,72	40
30	1,90	224,47	30	30	2,99	142,26	30
40	1,93	220,95	20	40	3,02	140,82	20
50	1,96	217,52	10	50	3,05	139,40	10
11 0	1,99	214,22	169 0	17 0	3,08	138,02	163 0
10	2,02	211,00	50	10	3,11	136,66	50
20	2,05	207,87	40	20	3,14	135,32	40
30	2,08	204,84	30	30	3,18	134,01	30
40	2,11	201,90	20	40	3,21	132,73	20
50	2,14	199,03	10	50	3,24	131,47	10
12 0	2,17	196,25	168 0	18 0	3,27	130,23	162 0
10	2,20	193,54	50	10	3,30	129,02	50
20	2,23	190,90	40	20	3,33	127,82	40
30	2,26	188,34	30	30	3,36	126,65	30
40	2,29	185,84	20	40	3,39	125,50	20
50	2,32	183,40	10	50	3,42	124,37	10
13 0	2,35	181,04	167 0	19 0	3,45	123,26	161 0
10	2,38	178,72	50	10	3,48	122,17	50
20	2,41	176,37	40	20	3,51	121,09	40
30	2,44	174,27	30	30	3,55	120,04	30
40	2,47	172,12	20	40	3,58	119,00	20
50	2,50	170,03	10	50	3,61	117,98	10
14 0	2,53	167,99	166 0	20 0	3,64	116,98	160 0
	Cot.	Tang.	D. M. Angle.		Cot.	Tang.	D.

Angl. D.	M.	Tang.	Cotang.	D.	M.	Angle. D.	M.	Tang.	Cotang.	D.	M.
		″	″					″	″		
20	0	3,64	116,98	160	0	26	0	4,76	89,35	154	0
	10	3,67	115,99		50		10	4,79	88,75		50
	20	3,70	115,02		40		20	4,82	88,17		40
	30	3,73	114,07		30		30	4,86	87,60		30
	40	3,76	113,13		20		40	4,89	87,03		20
	50	3,79	112,20		10		50	4,92	86,58		10
21	0	3,82	111,29	159	0	27	0	4,95	85,92	153	0
	10	3,85	110,39		50		10	4,98	85,37		50
	20	3,88	109,51		40		20	5,02	84,83		40
	30	3,92	108,64		30		30	5,05	84,30		30
	40	3,95	107,79		20		40	5,08	83,77		20
	50	3,98	106,94		10		50	5,11	83,25		10
22	0	4,01	106,11	158	0	28	0	5,14	82,74	152	0
	10	4,04	105,30		50		10	5,17	82,22		50
	20	4,07	104,49		40		20	5,21	81,71		40
	30	4,11	103,70		30		30	5,24	81,12		30
	40	4,14	102,91		20		40	5,27	80,73		20
	50	4,17	102,14		10		50	5,30	80,24		10
23	0	4,20	101,38	157	0	29	0	5,33	79,76	151	0
	10	4,23	100,63		50		10	5,36	79,28		50
	20	4,26	99,89		40		20	5,40	78,81		40
	30	4,29	99,17		30		30	5,43	78,34		30
	40	4,32	98,44		20		40	5,46	77,89		20
	50	4,35	97,74		10		50	5,49	77,43		10
24	0	4,38	97,04	156	0	30	0	5,53	76,98	150	0
	10	4,41	96,35		50		10	5,56	76,53		50
	20	4,45	95,67		40		20	5,59	76,09		40
	30	4,48	95,00		30		30	5,62	75,66		30
	40	4,51	94,34		20		40	5,66	75,23		20
	50	4,54	93,68		10		50	5,69	74,70		10
25	0	4,57	93,04	155	0	31	0	5,72	74,38	149	0
	10	4,60	92,40		50		10	5,75	73,96		50
	2	4,63	91,77		40		20	5,78	73,55		40
	30	4,67	91,16		30		30	5,82	73,14		30
	40	4,70	90,54		20		40	5,85	72,73		20
	50	4,73	89,94		10		50	5,88	72,33		10
26	0	4,76	89,35	154	0	32	0	5,91	71,93	148	0
		Cot.	Tang.	D. M. Angle.				Cot.	Tang.	D. M. Angle.	

Angl. D.	M.	Tang.	Cotang.			Angl. D.	M.	Tang.	Cotang.		
		″	″					″	″		
32	0	5,91	71,93	148	0	38	0	7,10	59,90	142	0
	10	5,94	71,54		50		10	7,13	59,62		50
	20	5,98	71,15		40		20	7,17	59,34		40
	30	6,01	70,77		30		30	7,21	59,06		30
	40	6,04	70,39		20		40	7,24	58,79		20
	50	6,07	70,01		10		50	7,27	58,52		10
33	0	6,11	69,63	147	0	39	0	7,30	58,25	141	0
	10	6,14	69,26		50		10	7,33	57,98		50
	20	6,18	68,90		40		20	7,37	57,71		40
	30	6,21	68,54		30		30	7,40	57,45		30
	40	6,24	68,16		20		40	7,44	57,19		20
	50	6,27	67,82		10		50	7,47	56,93		10
34	0	6,31	67,47	146	0	40	0	7,51	56,67	140	0
	10	6,34	67,12		50		10	7,54	56,42		50
	20	6,37	66,77		40		20	7,57	56,17		40
	30	6,41	66,43		30		30	7,61	55,92		30
	40	6,44	66,09		20		40	7,64	55,67		20
	50	6,47	65,95		10		50	7,68	55,42		10
35	0	6,50	65,42	145	0	41	0	7,71	55,17	139	0
	10	6,54	65,09		50		10	7,74	54,93		50
	20	6,57	64,76		40		20	7,78	54,69		40
	30	6,60	64,42		30		30	7,81	54,45		30
	40	6,64	64,12		20		40	7,85	54,21		20
	50	6,67	63,80		10		50	7,88	53,97		10
36	0	6,70	63,48	144	0	42	0	7,92	53,73	138	0
	10	6,73	63,17		50		10	7,95	53,50		50
	20	6,77	62,86		40		20	7,98	53,27		40
	30	6,80	62,55		30		30	8,02	53,04		30
	40	6,84	62,25		20		40	8,05	52,81		20
	50	6,87	61,95		10		50	8,08	52,58		10
37	0	6,90	61,65	143	0	43	0	8,12	52,36	137	0
	10	6,93	61,35		50		10	8,15	52,14		50
	20	6,97	61,06		40		20	8,19	51,92		40
	30	7,01	60,77		30		30	8,22	51,70		30
	40	7,04	60,48		20		40	8,26	51,48		20
	50	7,07	60,19		10		50	8,29	51,26		10
38	0	7,10	59,90	142	0	44	0	8,33	51,05	136	0
		Cot.	Tang.	D. M. Angle.				Cot.	Tang.	D. M. Angle.	

Angl. D.M.	Tang.	Cotang.		Angle. D.M.	Tang.	Cotang.	
	″	″			″	″	
44 0	8,33	51,05	136 0	50 0	9,62	44,23	130 0
10	8,36	50,84	50	10	9,65	44,06	50
20	8,40	50,63	40	20	9,69	43,90	40
30	8,43	50,42	30	30	9,73	43,73	30
40	8,47	50,21	20	40	9,76	43,57	20
50	8,50	50,00	10	50	9,80	43,40	10
45 0	8,54	49,80	135 0	51 0	9,84	43,24	129 0
10	8,57	49,59	50	10	9,87	43,08	50
20	8,61	49,39	40	20	9,91	42,92	40
30	8,65	49,19	30	30	9,95	42,76	30
40	8,68	48,99	20	40	9,98	42,60	20
50	8,72	48,79	10	50	10,02	42,44	10
46 0	8,76	48,59	134 0	52 0	10,06	42,29	128 0
10	8,79	48,39	50	10	10,09	42,13	50
20	8,83	48,20	40	20	10,13	41,98	40
30	8,87	48,01	30	30	10,17	41,82	30
40	8,90	47,82	20	40	10,20	41,67	20
50	8,94	47,63	10	50	10,24	41,52	10
47 0	8,97	47,44	133 0	53 0	10,28	41,37	127 0
10	9,00	47,25	50	10	10,31	41,22	50
20	9,04	47,06	40	20	10,35	41,07	40
30	9,07	46,88	30	30	10,39	40,92	30
40	9,11	46,69	20	40	10,43	40,77	20
50	9,14	46,51	10	50	10,47	40,62	10
48 0	9,18	46,33	132 0	54 0	10,51	40,48	126 0
10	9,21	46,15	50	10	10,54	40,33	50
20	9,25	45,97	40	20	10,58	40,19	40
30	9,29	45,79	30	30	10,62	40,04	30
40	9,32	45,61	20	40	10,66	39,90	20
50	9,36	45,43	10	50	10,70	39,76	10
49 0	9,40	45,26	131 0	55 0	10,74	39,62	125 0
10	9,43	45,08	50	10	10,77	39,48	50
20	9,47	44,91	40	20	10,81	39,34	40
30	9,51	44,74	30	30	10,85	39,20	30
40	9,54	44,57	20	40	10,89	39,06	20
50	9,58	44,40	10	50	10,93	38,92	10
50 0	9,62	44,23	130 0	56 0	10,97	38,79	124 0
	Cot.	Tang.	D.M. Angle.		Cot.	Tang.	D.M. Angle.

Angl. D. M.	Tang.	Cotang.		Angl. D. M.	Tang.	Cotang.	
	″	″			″	″	
56 0	10,97	38,79	124 0	62 0	12,39	34,33	118 0
10	11,00	38,65	50	10	12,43	34,21	50
20	11,04	38,52	40	20	12,47	34,10	40
30	11,08	38,39	30	30	12,51	33,99	30
40	11,12	38,25	20	40	12,56	33,88	20
50	11,16	38,12	10	50	12,60	33,77	10
57 0	11,20	37,99	123 0	63 0	12,64	33,66	117 0
10	11,23	37,86	50	10	12,68	33,55	50
20	11,27	37,73	40	20	12,72	33,44	40
30	11,31	37,60	30	30	12,76	33,33	30
40	11,35	37,47	20	40	12,80	33,22	20
50	11,39	37,34	10	50	12,84	33,11	10
58 0	11,43	37,21	122 0	64 0	12,89	33,01	116 0
10	11,47	37,08	50	10	12,93	32,90	50
20	11,51	36,96	40	20	12,97	32,80	40
30	11,55	36,83	30	30	13,01	32,69	30
40	11,59	36,71	20	40	13,05	32,58	20
50	11,63	36,58	10	50	13,09	32,48	10
59 0	11,67	36,46	121 0	65 0	13,14	32,38	115 0
10	11,71	36,33	50	10	13,18	32,28	50
20	11,75	36,21	40	20	13,22	32,17	40
30	11,79	36,09	30	30	13,26	32,07	30
40	11,83	35,97	20	40	13,30	31,97	20
50	11,87	35,85	10	50	13,34	31,87	10
60 0	11,91	35,73	120 0	66 0	13,39	31,76	114 0
10	11,95	35,61	50	10	13,43	31,66	50
20	11,99	35,49	40	20	13,47	31,56	40
30	12,03	35,37	30	30	13,52	31,46	30
40	12,07	35,25	20	40	13,56	31,36	20
50	12,11	35,13	10	50	13,60	31,26	10
61 0	12,15	35,02	119 0	67 0	13,65	31,16	113 0
10	12,19	34,90	50	10	13,69	31,06	50
20	12,23	34,79	40	20	13,73	30,96	40
30	12,27	34,67	30	30	13,78	30,87	30
40	12,31	34,56	20	40	13,82	30,77	20
50	12,35	34,44	10	50	13,86	30,67	10
62 0	12,39	34,33	118 0	68 0	13,91	30,58	112 0
	Cot.	Tang.	D. M. Angle.		Cot.	Tang.	D. M. Angle.

Angl. D. M.	Tang.	Cotang.		Angl. D. M.	Tang.	Cotang.	
	"	"			"	"	
68 0	13,91	30,58	112 0	74 0	15,54	27,37	106 0
10	13,95	30,48	50	10	15,58	27,28	50
20	13,99	30,39	40	20	15,63	27,20	40
30	14,04	30,29	30	30	15,68	27,12	30
40	14,08	30,20	20	40	15,73	27,04	20
50	14,12	30,10	10	50	15,78	26,96	10
69 0	14,17	30,01	111 0	75 0	15,83	26,88	105 0
10	14,21	29,91	50	10	15,87	26,80	50
20	14,26	29,82	40	20	15,92	26,72	40
30	14,30	29,73	30	30	15,97	26,64	30
40	14,35	29,64	20	40	16,01	26,56	20
50	14,39	29,55	10	50	16,06	26,48	10
70 0	14,44	29,46	110 0	76 0	16,11	26,40	104 0
10	14,48	29,37	50	10	16,16	26,32	50
20	14,53	29,28	40	20	16,21	26,24	40
30	14,57	29,19	30	30	16,26	26,16	30
40	14,62	29,10	20	40	16,31	26,08	20
50	14,66	29,01	10	50	16,36	26,00	10
71 0	14,71	28,92	109 0	77 0	16,41	25,93	103 0
10	14,75	28,83	50	10	16,45	25,85	50
20	14,80	28,74	40	20	16,50	25,77	40
30	14,85	28,65	30	30	16,55	25,70	30
40	14,89	28,56	20	40	16,60	25,62	20
50	14,94	28,47	10	50	16,65	25,54	10
72 0	14,99	28,39	108 0	78 0	16,70	25,47	102 0
10	15,03	28,30	50	10	16,75	25,39	50
20	15,08	28,21	40	20	16,80	25,32	40
30	15,12	28,13	30	30	16,85	25,24	30
40	15,17	28,04	20	40	16,90	25,17	20
50	15,21	27,95	10	50	16,95	25,09	10
73 0	15,26	27,87	107 0	79 0	17,00	25,02	101 0
10	15,30	27,78	50	10	17,05	24,94	50
20	15,35	27,70	40	20	17,10	24,87	40
30	15,40	27,62	30	30	17,15	24,80	30
40	15,44	27,53	20	40	17,20	24,72	20
50	15,49	27,45	10	50	17,25	24,65	10
74 0	15,54	27,37	106 0	80 0	17,31	24,58	100 0
	Cot.	Tang.	D.M. Angle.		Cot.	Tang.	D. M. Angle.

Angl. D.M.	Tang.	Cotang.		Angle. D.M.	Tang.	Cotang.	
	"	"			"	"	
80 0	17,31	24,58	100 0	85 0	18,90	22,50	95 0
10	17,36	24,50	50	10	18,95	22,43	50
20	17,41	24,43	40	20	19,01	22,37	40
30	17,46	24,36	30	30	19,06	22,30	30
40	17,51	24,29	20	40	19,12	22,24	20
50	17,56	24,22	10	50	19,17	22,18	10
81 0	17,62	24,15	99 0	86 0	19,23	22,12	94 0
10	17,67	24,08	50	10	19,28	22,05	50
20	17,72	24,01	40	20	19,34	21,99	40
30	17,77	23,94	30	30	19,40	21,92	30
40	17,82	23,87	20	40	19,45	21,86	20
50	17,87	23,80	10	50	19,51	21,80	10
82 0	17,93	23,73	98 0	87 0	19,56	21,74	93 0
10	17,98	23,66	50	10	19,62	21,67	50
20	18,03	23,59	40	20	19,68	21,61	40
30	18,09	23,52	30	30	19,74	21,54	30
40	18,14	23,45	20	40	19,80	21,48	20
50	18,19	23,38	10	50	19,86	21,42	10
83 0	18,25	23,31	97 0	88 0	19,92	21,36	92 0
10	18,30	23,24	50	10	19,97	21,29	50
20	18,35	23,17	40	20	20,03	21,23	40
30	18,41	23,11	30	30	20,09	21,17	30
40	18,46	23,04	20	40	20,15	21,11	20
50	18,51	22,97	10	50	20,21	21,05	10
84 0	18,57	22,91	96 0	89 0	20,27	20,99	91 0
10	18,62	22,84	50	10	20,33	20,93	50
20	18,68	22,77	40	20	20,39	20,87	40
30	18,73	22,70	0	30	20,45	20,81	30
40	18,79	22,63	0	40	20,51	20,75	20
50	18,84	22,56	10	50	20,57	20,69	10
85 0	18,90	22,50	95 0	90 0	20,63	20,63	90 0
	Cot.	Tang.	D.M. Angle.		Cot.	Tang.	D.M. Angle.

In the reduction to the horizon, the tangent number is positive and the co-tangent is negative. The contrary takes place in reducing the horizontal angle to that of the chords.

TA

Product sec.

		3°. 0′	2°. 50′	2°. 40′	2°. 30′	2°. 20′	2°. 10′	2°. 0′	1°. 50
3°	0	1,0027	1,0026	1,0025	1,0023	1,0022	1,0021	1,0020	1,001
2.	50	1,0026	1,0025	1,0023	1,0021	1,0020	1,0019	1,0018	1,001
	40	1,0025	1,0024	1,0022	1,0020	1,0019	1,0018	1,0017	1,001
	30	1,0023	1,0022	1,0020	1,0019	1,0018	1,0017	1,0016	1,001
	20	1,0022	1,0021	1,0019	1,0018	1,0017	1,0015	1,0014	1,00
	10	1,0021	1,0019	1,0018	1,0017	1,0015	1,0014	1,0013	1,00
2.	0	1,0020	1,0018	1,0017	1,0016	1,0014	1,0013	1,0012	1,00
1.	50	1,0018	1,0017	1,0016	1,0015	1,0013	1,0012	1,0011	1,00
	40	1,0018	1,0016	1,0015	1,0014	1,0012	1,0011	1,0010	1,00
	30	1,0017	1,0016	1,0014	1,0013	1,0012	1,0011	1,0009	1,00
	20	1,0016	1,0015	1,0014	1,0012	1,0011	1,0010	1,0009	1,00
	10	1,0016	1,0014	1,0013	1,0012	1,0010	1,0009	1,0008	1,00
1.	0	1,0015	1,0014	1,0012	1,0011	1,0010	1,0009	1,0008	1,00
0.	50	1,00 5	1,0013	1,0012	1,0011	1,0009	1,0008	1,0007	1,00
	40	1,0015	1,0013	1,0011	1,0011	1,0009	1,0008	1,0007	1,00
	30	1,0014	1,0013	1,0011	1,0010	1,0009	1,0007	1,0006	1,00
	20	1,0014	1,0012	1,0011	1,0010	1,0008	1,0007	1,0006	1,00
	10	1,0014	1,0012	1,0011	1,0010	1,0008	1,0007	1,0006	1,00
0.	0	1,0014	1,0012	1,0011	1,0010	1,0008	1,0007	1,0006	1,0

I.

ments H and *h*.

30′	1°. 20′	1°. 10′	1°. 0′	0°. 50′	0°. 40′	0°. 30′	0°. 20′	0°. 10′	0°. 0′
017	1,0016	1,0016	1,0015	1,0015	1,0014	1,0014	1,0014	1,0014	1,0014
016	1,0015	1,0014	1,0014	1,0013	1,0013	1,0013	1,0012	1,0012	1,0012
014	1,0013	1,0012	1,0012	1,0012	1,0012	1,0012	1,0011	1,0011	1,0011
013	1,0012	1,0011	1,0011	1,0011	1,0010	1,0010	1,0010	1,0010	1,0010
012	1,0011	1,0010	1,0010	1,0009	1,0009	1,0009	1,0009	1,0008	1,0008
011	1,0010	1,0009	1,0009	1,0008	1,0008	1,0007	1,0007	1,0007	1,0007
010	1,0009	1,0008	1,0008	1,0007	1,0007	1,0006	1,0006	1,0006	1,0006
009	1,0008	1,0007	1,0007	1,0006	1,0006	1,0006	1,0005	1,0005	1,0005
008	1,0007	1,0006	1,0006	1,0005	1,0005	1,0005	1,0004	1,0004	1,0004
007	1,0006	1,0006	1,0005	1,0005	1,0004	1,0004	1,0004	1,0003	1,0003
006	1,0005	1,0005	1,0004	1,0004	1,0003	1,0003	1,0003	1,0002	1,0003
006	1,0005	1,0004	1,0004	1,0003	1,0003	1,0002	1,0002	1,0002	1,0002
005	1,0004	1,0004	1,0003	1,0002	1,0002	1,0002	1,0002	1,0002	1,0002
005	1,0004	1,0003	1,0003	1,0002	1,0002	1,0002	1,0002	1,0002	1,0001
004	1,0003	1,0003	1,0002	1,0002	1,0002	1,0002	1,0002	1,0001	1,0001
004	1,0003	1,0002	1,0002	1,0002	1,0002	1,0002	1,0001	1,0001	1,0000
004	1,0003	1,0002	1,0002	1,0002	1,0002	1,0001	1,0001	1,0001	1,0000
003	1,0003	1,0002	1,0002	1,0002	1,0002	1,0001	1,0001	1,0000	1,0000
003	1,0003	1,0002	1,0002	1,0001	1,0002	1,0000	1,0000	1,0000	1,0000

TABLE IV.

ARGUMENT *A, or Observed Angle.*

A.	—	A	+
°	''	°	''
3	0,458	93	0,001
6	0,230	96	0,003
9	0,153	99	0,004
12	0,114	102	0,005
15	0,090	105	0,007
18	0,075	108	0,008
21	0,063	111	0,009
24	0,054	114	0,011
27	0,048	117	0,012
30	0,042	120	0,014
33	0,037	123	0,016
36	0,033	126	0,018
39	0,030	129	0,020
42	0,027	132	0,022
45	0,024	135	0,024
48	0,022	138	0,027
51	0,020	141	0,030
54	0,018	144	0,033
57	0,016	147	0,037
60	0,014	150	0,042
63	0,012	153	0,048
66	0,011	156	0,054
69	0,009	159	0,063
72	0,008	162	0,075
75	0,006	165	0,091
78	0,005	168	0,114
81	0,004	171	0,153
84	0,003	174	0,231
87	0,001	177	0,467
90	0,000	180	0,000

This table contains the two small terms $-\frac{1}{2}(n \text{ sec. H sec. } h)^2 \frac{\text{cot. A}}{\text{sin. } 1''} + \frac{1}{2}(n \text{ sec. H sec. } h)^3 \left(\frac{\frac{1}{3}+\text{cot}^2\text{A}}{\text{sin.}^2\ 1''}\right)$ of the formula for the reduction to the horizon. (see page 79).

It supposes n sec. H sec. $h=100''$; and it appears that with this value these terms are still insensible, and may almost always be neglected. The numbers in this table increase or diminish in the ratio of the squares of the values of (n sec. H sec. h); thus for 120'' they are to be multiplied by $\left(\frac{120}{100}\right)^2=(1{\cdot}20)^2=1.44$, and in general by $\left(\frac{100+x}{100}\right)^2$.

It also appears from a comparison of the two columns of the table, that the part which depends upon the cube is nearly nothing.

TABLES

FOR

REDUCING TO THE MERIDIAN

THE

Zenith Distances of the Heavenly Bodies,

OBSERVED WITH BORDA'S CIRCLE,

OR OTHER INSTRUMENTS.

TABLE I.

Reduction to the Meridian for the Observations.

ARGUMENT. *Horary Angle in Time.*

Sec.	0′	1′	2′	3′	4′	5′	6′	7′
0	0″0	2″0	7″8	17″7	31″4	49″1	70″7	96″2
1	0,0	2,0	8,0	17,9	31,7	49,4	71,1	96,9
2	0,0	2,1	8,1	18,1	31,9	49,7	71,5	97,1
3	0,0	2,2	8,2	18,3	32,2	50,1	71,9	97,6
4	0,0	2,2	8,4	18,5	32,5	50,4	72,3	98,1
5	0,0	2,3	8,5	18,7	32,7	50,7	72,7	98,5
6	0,0	2,4	8,7	18,9	33,0	51,1	73,1	99,0
7	0,0	2,4	8,8	19,1	33,3	51,4	73,5	99,4
8	0,0	2,5	8,9	19,3	33,5	51,7	73,9	99,9
9	0,0	2,6	9,1	19,5	33,8	52,1	74,3	100,4
10	0,1	2,7	9,2	19,7	34,1	52,4	74,7	100,8
11	0,1	2,7	9,4	19,9	34,4	52,7	75,1	101,3
12	0,1	2,8	9,5	20,1	34,6	53,1	75,5	101,8
13	0,1	2,9	9,6	20,3	34,9	53,4	75,9	102,3
14	0,1	3,0	9,8	20,5	35,2	53,8	76,3	102,7
15	0,1	3,1	9,9	20,7	35,5	54,1	76,7	103,2
16	0,1	3,1	10,1	20,9	35,7	54,5	77,1	103,7
17	0,2	3,2	10,2	21,2	36,0	54,8	77,5	104,2
18	0,2	3,3	10,4	21,4	36,3	55,1	77,9	104,6
19	0,2	3,4	10,5	21,6	36,6	55,5	78,3	105,1
20	0,2	3,5	10,7	21,8	36,9	55,8	78,8	105,6
21	0,3	3,6	10,8	22,0	37,2	56,2	79,2	106,0
22	0,3	3,7	11,0	22,3	37,4	56,5	79,6	106,6
23	0,3	3,8	11,1	22,5	37,7	56,9	80,0	107,0
24	0,3	3,8	11,3	22,7	38,0	57,3	80,4	107,5
25	0,3	3,9	11,5	22,9	38,2	57,6	80,8	108,0
26	0,4	4,0	11,6	23,1	38,6	58,0	81,3	108,5
27	0,4	4,1	11,8	23,4	38,9	58,3	81,7	109,0
28	0,4	4,2	11,9	23,6	39,2	58,7	82,1	109,5
29	0,5	4,3	12,1	23,8	39,5	59,0	82,5	110,0

Continuation of TABLE I.

ARGUMENT, *Horary Angle in Time.*

Sec.	0′	1′	2′	3′	4′	5′	6′	7′
	″	″	″	″	″	″	″	″
30	0,5	4,4	12,3	24,0	39,8	59,4	83,0	110,4
31	0,5	4,5	12,4	24,3	40,1	59,8	83,4	110,9
32	0,6	4,6	12,6	24,5	40,3	60,1	83,8	111,4
33	0,6	4,7	12,8	24,7	40,6	60,5	84,2	111,9
34	0,6	4,8	12,9	25,0	40,9	60,8	84,7	112,4
35	0,7	4,9	13,1	25,2	41,2	61,2	85,1	112,9
36	0,7	5,0	13,3	25,4	41,5	61,6	85,5	113,4
37	0,7	5,1	13,4	25,7	41,8	61,9	86,0	113,9
38	0,8	5,2	13,6	25,9	42,1	62,3	86,4	114,4
39	0,8	5,3	13,8	26,2	42,5	62,7	86,8	114,9
40	0,9	5,4	14,0	26,4	42,8	63,0	87,3	115,4
41	0,9	5,6	14,1	26,6	43,1	63,4	87,7	115,9
42	1,0	5,7	14,3	26,9	43,4	63,8	88,1	116,4
43	1,0	5,8	14,5	27,1	43,7	64,2	88,6	116,9
44	1,1	5,9	14,7	27,4	44,0	64,5	89,0	117,4
45	1,1	6,0	14,8	27,6	44,3	64,9	89,5	117,9
46	1,2	6,1	15,0	27,9	44,6	65,3	89,9	118,4
47	1,2	6,2	15,2	28,1	44,9	65,7	90,3	118,9
48	1,3	6,4	15,4	28,3	45,2	66,0	90,8	119,5
49	1,3	6,5	15,6	28,6	45,5	66,4	91,2	120,0
50	1,4	6,6	15,8	28,8	45,9	66,8	91,7	120,5
51	1,4	6,7	15,9	29,1	46,2	67,2	92,1	121,0
52	1,5	6,8	16,1	29,4	46,5	67,6	92,6	121,5
53	1,5	7,0	16,3	29,6	46,8	68,0	93,0	122,0
54	1,6	7,1	16,5	29,9	47,1	68,3	93,5	122,5
55	1,6	7,2	16,7	30,1	47,5	68,7	93,9	123,1
56	1,7	7,3	16,9	30,4	47,8	69,1	94,4	123,6
57	1,8	7,5	17,1	30,6	48,1	69,5	94,8	124,1
58	1,8	7,6	17,3	30,9	48,4	69,9	95,3	124,6
59	1,9	7,7	17,5	31,1	48,8	70,3	95,7	125,1

Continuation of TABLE I.

ARGUMENT, *Horary Angle in Time.*

Sec.	8′	9′	10′	11′	12′	13′	14′	15′
	″	″	″	″	″	″	″	″
0	125,7	159,0	196,3	237,5	280,7	331,8	384,7	441,6
1	126,2	159,6	197,0	238,3	283,5	332,6	385,6	442,6
2	126,7	160,2	197,6	239,0	284,2	333,4	386,5	443,6
3	127,2	160,8	198,3	239,7	285,0	334,3	387,5	444,6
4	127,8	161,4	198,9	240,4	285,8	335,3	388,4	445,6
5	128,3	162,0	199,6	241,2	286,6	336,0	389,3	446,5
6	128,8	162,6	200,3	241,9	287,4	336,9	390,2	447,5
7	129,4	163,2	200,9	242,6	288,2	337,7	391,1	448,5
8	129,9	163,8	201,6	243,3	289,0	338,6	392,1	449,5
9	130,4	164,4	202,2	244,1	289,8	339,4	393,0	450,5
10	131,0	165,0	202,9	244,8	290,6	340,3	393,9	451,5
11	131,5	165,6	203,6	245,5	291,4	341,2	394,8	452,5
12	132,0	166,2	204,2	246,2	292,2	342,0	395,8	453,5
13	132,6	166,8	204,9	247,0	293,0	342,9	396,7	454,5
14	133,1	167,4	205,6	247,7	293,8	343,7	397,6	455,5
15	133,6	168,0	206,3	248,5	294,6	344,6	398,6	456,5
16	134,2	168,6	206,9	249,2	295,4	345,5	399,5	457,5
17	134,7	169,2	207,6	249,9	296,2	346,3	400,5	458,5
18	135,3	169,8	208,3	250,7	297,0	347,2	401,4	459,5
19	135,8	170,4	208,9	251,4	297,8	348,1	402,3	460,5
20	136,4	171,0	209,6	252,2	298,6	349,0	403,3	461,5
21	136,9	171,6	210,3	252,9	299,4	349,8	404,2	462,5
22	137,4	172,2	211,0	253,6	300,2	350,7	405,1	463,5
23	138,0	172,9	211,6	254,4	301,0	351,6	406,1	464,5
24	138,5	173,5	212,3	255,1	301,8	352,5	407,0	465,5
25	139,1	174,1	213,0	255,9	302,6	353,3	408,0	466,5
26	139,6	174,7	213,7	256,6	303,5	354,2	408,9	467,5
27	140,2	175,3	214,4	257,4	304,3	355,1	409,9	468,5
28	140,7	175,9	215,1	258,1	305,1	356,0	410,8	469,5
29	141,3	176,6	215,8	258,9	305,9	356,9	411,7	470,5

Continuation of TABLE I.

ARGUMENT, *Horary Angle in Time.*

Sec.	8′	9′	10′	11′	12′	13′	14′	15′
	″	″	″	″	″	″	″	″
30	141,8	177,2	216,4	259,6	306,7	357,7	412,7	471,5
31	142,4	177,8	217,1	260,4	307,5	358,6	413,6	472,6
32	143,0	178,4	217,8	261,1	308,4	359,5	414,6	473,6
33	143,5	179,0	218,5	261,9	309,2	360,5	415,6	474,6
34	144,1	179,7	219,2	262,6	310,0	361,1	416,6	475 6
35	144,6	180,3	219,9	263,4	310,8	362,2	417,5	476,6
36	145,2	180,9	220,6	264,1	311,6	363,1	418,4	477,6
37	145,8	181,6	221,3	264,9	312,5	363,9	419,4	478,7
38	146,3	182,2	222,0	265,7	313,3	364,6	420,3	479,7
39	146,9	182,8	222,7	266,4	314,2	365,7	421,3	480,7
40	147,5	183,4	223,4	267,2	315,0	366,6	422,2	481,7
41	148,0	184,1	224,1	267,9	315,8	367,5	423,2	482,8
42	148,6	184,7	224,8	268,7	316,6	368,4	424,2	483,8
43	149,2	185,4	225,5	269,5	317,4	369,3	425,1	484,8
44	149,7	186,0	226,2	270,2	318,3	370,2	426,1	485,8
45	150,3	186,6	226,9	271,0	319,1	371,1	427,0	486,9
46	150,9	187,3	227,6	271,8	319,9	372,0	428,0	487,9
47	151,5	187,9	228,3	272,6	320,8	372,9	429,0	488,9
48	152,0	188,5	229,0	273,3	321,6	373,8	430,0	490,0
49	152,6	189,2	229,7	274,1	322,4	374,7	430,9	491,0
50	153,2	189,8	230,4	274,9	323,3	375,6	431,9	492,0
51	153,8	190,5	231,1	275,6	324,1	376,5	432,8	493,1
52	154,4	191,1	231,8	276,4	325,0	377,4	433,8	494,1
53	154,9	191,8	232,5	277,2	325,8	378,3	434,8	495,2
54	155,5	192,4	233,3	278,0	326,7	379,2	435,7	496,2
55	156,1	193,1	234,0	278,9	327,5	380,2	436,7	497,2
56	156,7	193,7	234,7	279,5	328,4	381,1	437,7	498,2
57	157,3	194,4	235,4	280,3	329,2	382,0	438,7	499,2
58	157,8	195,0	236,1	281,1	330,0	382,9	439,6	500,3
59	158,4	195,7	236,8	281,9	330,9	383,8	440,6	501,4

TABLE II.

For the Latitude 48° 51′. Factor F for multiplying the numbers of Table 1.

ARGUMENT, *Declination.*

D	F.	*Dif.*	D.	F.	*Dif.*	D.	F.	*Dif.*
41. S	0.4969		10. S	0.7572		21. N	1.3150	
40.	0.5042	75	9.	0.7676	104	22.	1.3509	359
39.	0.5118	76	8.	0.7783	107	23.	1.3893	384
38.	0.5193	75	7.	0.7892	109	24.	1.4305	412
37.	0.5269	76	6.	0.8004	112	25.	1.4750	435
36.	0.5345	76	5.	0.8118	114	26.	1.5231	481
35.	0.5422	77	4.	0.8236	118	27.	1.5754	523
34.	0.5498	76	3.	0.8356	120	28.	1.6324	570
33.	0.5575	77	2.	0.8480	124	29.	1.6950	626
32.	0.5652	77	1. S	0.8608	128	30.	1.7636	688
31.	0.5730	78	0.	0.8739	131	31.	1.8401	763
30.	0.5808	78	1. N	0.8874	135	32.	1.9252	851
29.	0.5887	79	2.	0.9014	140	33.	2.0206	954
28.	0.5967	80	3.	0.9158	144	34.	2.1286	1080
27.	0.6047	80	4.	0.9308	150	35.	2.2517	1231
26.	0.6127	80	5.	0.9463	155	36.	2.3938	1421
25.	0.6209	82	6.	0.9623	160	37.	2.5592	1654
24.	0.6291	82	7.	0.9789	166	38.	2.7547	1955
23.	0.6375	84	8.	0.9963	174	39.	2.9894	2347
22.	0.6459	84	9.	1.0143	180	40.	3.2765	2871
21.	0.6544	85	10.	1.0331	188	41.	3.6361	3596
20.	0.6630	86	11.	1.0527	196	42.	4.1038	4677
19.	0.6718	88	12.	1.0732	205	43.	4.7217	6179
18.	0.6807	89	13.	1.0947	215	44.	5.5986	8769
17.	0.6897	90	14.	1.1174	227	45.	6.9298	
16.	0.6988	91	15.	1.1411	237	46.	9.1935	
15.	0.7081	93	16.	1.1661	250	47.	13.9014	
14.	0.7175	94	17.	1.1925	264	48.	29.90	
13.	0.7272	97	18.	1.2205	280	49.	16.49	
12.	0.7371	99	19.	1.2501	296			
11.	0.7470	99	20.	1.2815	314			
		102			335			

Continuation of TABLE II.

ARGUMENT, *Declination.*

D.	F.	*Dif.*	D.	F.	*Dif.*	D.	F.	*Dif.*
50. N.	21.075		80. N.	0.2209	274	70. N.	0.2569	98
51.	11.039		81.	0.1935	260	69.	0.2667	96
52.	7.373		82.	0.1675	246	68.	0 2763	94
53.	5.472		83.	0.1429	234	67.	0.2857	92
54.	4.309		84.	0.1195	223	66.	0 2949	91
55.	3.523		85.	0.0972	212	65.	0 3040	90
56.	2.9563		86.	0.0760	202	64.	0.3130	89
57.	2.5281		87.	0 0558	194	63.	0.3219	87
58.	2.1929		88.	0.0364	186	62.	0.3306	86
59.	1.9232		89.	0.0178	178	61.	0.3392	85
60.	1.7014		90.	0.0000	171	60.	0.3477	84
61.	1.5158		89.	0.0171	165	59.	0.3561	83
62.	1.3580		88.	0.0336	158	58.	0.3644	82
63.	1 2220		87.	0.0494	153	57.	0.3726	81
64.	1.1037		86.	0.0647	148	56.	0.3807	80
65.	0 9998	922	85.	0.0795	143	55.	0.3887	80
66.	0.9076	822	84.	0.0938	139	54.	0.3967	79
67.	0.8254	740	83.	0.1077	134	53.	0.4046	79
68.	0.7514	668	82.	0.1211	130	52.	0.4125	78
69.	0.6846	608	81.	0.1341	126	51.	0.4203	78
70.	0.6238	552	80.	0.1467	123	50.	0.4281	77
71.	0.5686	514	79.	0.1590	120	49.	0 4358	77
72.	0.5172	470	78.	0.1710	116	48.	0.4435	76
73.	0.4702	434	77.	0.1826	114	47.	0 4511	76
74.	0.4268	404	76.	0.1940	111	46.	0 4587	77
75.	0.3864	375	75.	0.2051	108	45.	0.4664	76
76.	0.3489	351	74.	0.2159	105	44.	0.4740	75
77.	0.3138	329	73.	0.2264	104	43.	0.4815	76
78.	0.2809	309	72.	0.2368	102	42.	0.4891	
79.	0.2500	291	71.	0.2470	99			
80.	0.2209		70.	0.2569				

III. GENERAL TABLE. *Second Term.*

ARGUMENT, *Horary Angle.*

M.S.		S.	Dif.	M.S.		S.	Dif.	M.S.		S.	Dif.
0.	0	0.000		8.	10	0.041		12.	10	0.205	
1.	0	0.000			20	0.045	4		20	0.217	12
2.	0	0.000			30	0.049	4		30	0.229	12
3.	0	0.001			40	0.053	4		40	0.241	12
4.	0	0.002			50	0.057	4		50	0.254	13
5.	0	0.006		9.	0	0.061	4	13.	0	0.267	13
	10	0.007			10	0.066	5		10	0.281	14
	20	0.008			20	0.071	5		20	0.295	14
	30	0.009			30	0.076	5		30	0.310	15
	40	0.010			40	0.081	5		40	0.326	16
	50	0.011			50	0.087	6		50	0.342	16
6.	0	0.012		10.	0	0.093	6	14.	0	0.359	17
	10	0.013			10	0.100	7		10	0.376	17
	20	0.014			20	0.107	7		20	0.394	18
	30	0.016			30	0.114	7		30	0.413	19
	40	0.018			40	0.121	7		40	0.432	19
	50	0.020			50	0.129	8		50	0.452	20
7.	0	0.022		11.	0	0.137	8	15.	0	0.473	21
	10	0.024			10	0.145	8		10	0.494	21
	20	0.026			20	0.154	9		20	0.516	22
	30	0.029			30	0.163	9		30	0.539	23
	40	0.032			40	0.173	10		40	0.563	24
	50	0.035			50	0.183	10		50	0.587	24
8.	0	0.038		12.	0	0.194	11	16.	0	0.612	25

This second term is always additive, but the first term is additive only in the inferior passages of the circumpolar stars

TABLE IV.

Factors for multiplying the numbers in Table III.

ARGUMENT, Declination.

D.	f.	Dif.	D.	f.	Dif.	D.	f.	Dif
41. S.	0.001	4	10. S.	0.347	23	20. N.	2.974	299
40.	0.005	5	9.	0.370	26	21.	3.273	332
39.	0.010	5	8.	0.396	27	22.	3.605	379
38.	0.015	5	7.	0.423	28	23.	3.984	434
37.	0.020	6	6.	0.451	31	24.	4.418	503
36.	0.026	6	5.	0.482	32	25.	4.921	584
35.	0.032	6	4.	0.514	35	26.	5.505	684
34.	0.038	6	3.	0.549	37	27.	6.189	807
33.	0.044	7	2.	0.586	39	28.	6.996	962
32.	0.051	8	1. S.	0.625	42	29.	7.958	1154
31.	0.059	8	0.	0.667	46	30.	9.112	
30.	0.067	8	1. N.	0.713	49	31.	10.515	
29.	0.075	8	2.	0.762	52	32.	12.237	
28.	0.083	9	3.	0.814	57	33.	14.381	
27,	0.092	10	4.	0.871	61	34.	17.088	
26.	0.102	10	5.	0.932	67	35.	20.566	
25.	0.112	10	6.	0.999	71	36.	25.120	
24.	0.122	11	7.	1.070	78	37.	31.216	
23.	0. 33	12	8.	1.148	85	38.	39.591	
22.	0.145	12	9.	1.233	92	39.	51.468	
21.	0.157	13	10.	1.325	101	40.	68.949	
20.	0.170	14	11.	1.426	111	41.	95.90	
19.	0.184	14	12.	1.537	122	42.	140.19	
18.	0.198	15	13.	1.659	134	43.	217.60	
17.	0.213	16	14.	1.793	149	44.	369.40	
16.	0.229	17	15.	1.942	164	45.	713.59	
15.	0.246	18	16.	2.106	183	46.	1697.8	
14.	0.264	19	17.	2.289	205	47.	5983.0	
13.	0.283	20	18.	2.494	229	48.		
12.	0.303	21	19.	2.723	251	49.		
11.	0.324		20.	2.974				

Continuation of TABLE IV.

ARGUMENT, Declination.

D.	*f.*	*Dif.*	D.	*f.*	*Dif.*	D.	*f.*	*Dif.*
50. N.	......		80. N.	0.080	20	70. N.	0.036	2
51.	324.6		81.	0.060	17	69.	0.038	1
52.	987.7		82.	0.043	13	68.	0.039	1
53.	412.7		83.	0.030	10	67.	0.040	0
54.	206.0		84.	0.020	7	66.	0.040	1
55.	115.2		85.	0.013	5	65.	0.041	0
56.	69.67		86.	0.008	4	64.	0.041	0
57.	44.63		87.	0.004	2	63.	0.041	1
58.	29.86		88.	0.002	0	62.	0.042	0
59.	20.66		89.	0.000		61.	0.042	1
60.	15.03		90.			60.	0.041	0
61.	10.67		89.	inferior		59.	0.041	1
62.	7.89		88.	Passag.	2	58.	0.040	1
63.	5.92		87.	0.002	2	57.	0.039	1
64.	4.50		86.	0.004	2	56.	0.038	1
65.	3.45	88	85.	0.006	2	55.	0.037	1
66.	2.67	59	84.	0.008	2	54.	0.036	2
67.	2.08	45	83.	0.010	3	53.	0.034	1
68.	1.63	35	82.	0.013	2	52.	0.033	2
69.	1.28	27	81.	0.015	2	51.	0.031	2
70.	1.01	21	80.	0.017	3	50.	0.029	3
71.	0.80	17	79.	0.020	2	49.	0.026	2
72.	0.63	14	78.	0.022	2	48.	0.024	3
73.	0.49	10	77.	0.024	2	47.	0.021	3
74.	0.39	9	76.	0.026	2	46.	0.018	3
75.	0.30	6	75.	0.028	2	45.	0.015	4
76.	0.24		74.	0.030	2	44.	0.011	3
77.	0.18	6	73.	0.032	2	43.	0.008	4
78.	0.14	4	72.	0.034	1	42.	0.004	
79.	0.10	4	71.	0.035	1			
80.	0.08	2	70.	0.036				

TABLE V. *Latitude* 48° 51′.

Horary Angle when the Reduction varies one second of a degree for each second of Time.

D.	Horary Angle.	*Dif.*	D.	Horary Angle.	*Dif.*	D.	Horary Angle.	*Dif.*
41. S.	30′,9	5	10. S.	20′,2	2	20. N.	11′,9	2
40.	30, 4	5	9.	19, 9	2	21.	11,7	4
39.	29, 9	4	8.	19, 7	3	22.	11, 3	3
38.	29, 5	4	7.	19, 4	3	23.	11,0	3
37.	29, 1	4	6.	19, 1	3	24.	10,7	4
36.	28, 7	4	5.	18, 8	2	25.	10, 3	3
35.	28, 3	4	4.	18, 6	2	26.	10, 1	3
34.	27, 9	4	3.	18, 4	3	27.	9, 7	3
33.	27, 5	4	2.	18, 1	3	28.	9, 4	3
32.	27, 1	4	1. S.	17, 8	3	29.	9, 1	4
31.	26, 7	3	0.	17, 5	2	30.	8, 7	4
30.	26, 4	4	1. N.	17, 3	3	31.	8, 3	4
29.	26, 0	3	2.	17, 0	2	32.	7, 9	4
28.	25, 7	4	3.	16, 8	3	33.	7, 5	3
27.	25, 3	3	4.	16, 5	2	34.	7, 2	4
26.	25, 0	4	5.	16, 3	4	35.	6, 8	4
25.	24, 6	3	6.	15, 9	2	36.	6, 4	4
24.	24, 3	3	7.	15, 7	3	37.	6, 0	5
23.	24, 0	3	8.	15, 4	3	38.	5, 5	4
22.	23, 7	3	9.	15, 1	3	39.	5, 1	4
21.	23, 4	3	10.	14, 8	3	40.	4, 7	5
20.	23, 1	3	11.	14, 5	2	41.	4, 2	5
19.	22, 8	3	12.	14, 3	3	42.	3, 7	5
18.	22, 5	3	13.	14, 0	2	43.	3, 2	5
17.	22, 2	3	14.	13, 7	4	44.	2, 7	5
16.	21, 9	3	15.	13, 3	2	45.	2, 2	5
15.	21, 6	2	16.	13, 1	2	46.	1, 7	6
14.	21, 4	3	17.	12, 9	3	47.	1, 1	5
13.	21, 1	2	18.	12, 6	4	48.	0, 5	5
12.	20, 9	3	19.	12, 2	3	49.	0, 0	
11.	20, 6		20.	11, 9		50.		

Continuation of TABLE V.

Horary Angle when the Reduction varies one second of a degree for each second of Time.

D.	Horary Angle.	*Dif.*	D.	Horary Angle.	*Dif.*	D.	Horary Angle.	*Dif.*
50. N.	......		80. N.	70',2		70. N.	60',2	23
51.	1',4	7	81.	80,6		69.	57,9	20
52.	2,1	7	82.	93,8		68.	55,9	18
53.	2,8	7	83.	111,3		67.	54,1	18
54.	3,5	8	84.	135,7		66.	52,3	16
55.	4,3	9	85.	160,9		65.	50,7	15
56.	5 2	9	86.	245,2		64.	49,2	13
57.	6,1	9	87.			63.	47,9	14
58.	7,0	9	88.			62.	46,5	12
59.	7,9	11	89.			61.	45,3	11
60.	9,0	11	90.			60.	44,2	10
61.	10,1	12	89.			59.	43,2	10
62.	11,3	12	88.	inferior		58.	42,2	10
63.	12,5	14	87.	Passage		57.	41,2	9
64.	13,9	14	86.			56.	40,3	8
65.	15,3	16	85.			55.	39,5	8
66.	16,9	16	84.	181,1		54.	38,7	8
67.	18,5	18	83.	153,1		53.	37,9	7
68.	20,3	20	82.	133,7		52.	37,2	7
69.	22,3	22	81.	119,3		51.	36,5	6
70.	24,5	24	80.	108,1		50.	35,9	7
71.	26,9	28	79.	99,2	74	49.	35,2	6
72.	29,7	29	78.	91,8	61	48.	34,6	6
73.	32,6	30	77.	85,7	53	47.	34,0	5
74.	35,6	31	76.	80,4	45	46.	33,5	6
75.	38,7	34	75.	75,9	40	45.	32,9	6
76.	44,1	50	74.	71,9	34	44.	32,3	5
77.	49,1	58	73.	68,5	31	43.	31,8	5
78.	54,9	70	72.	65,4	27	42.	31,3	
79.	61,9	83	71.	62,7	25			
80.	70,2		70.	60,2				

EXAMPLE.

Horary Angles.	Tab. I.	Table III.
15′ 57″	499″2	0″60
14. 23	406,1	0,40
12. 59	330,9	0,27
11. 32	261,1	0,16
5....10. 1	197,0	0,09
8. 34	144,1	0,05
7. 8	99,9	0,02
6. 10	74,7	0,01
5. 15	54,1	0,01
10.... 3. 52	29,4	0,00
2. 23	11,1	0,00
2. 4	8,4	0,00
3. 40	26,4	0,00
5. 1	49,4	0,01
15.... 6. 39	86,8	0,02
7. 58	124,6	0,04
9. 27	175,3	0,07
11. 3	239,7	0,14
12. 7	288,2	0,21
20....13. 1	332,6	0,27
14. 27	409,9	0,41
15. 34	475,6	0,55

$D = 33° 24' S$
$= 33°.4$

Table II.

For 33° S, $F = 0.5575$
For 0,4 —31

$F = 0.5544$

Table IV.

$f = 0{,}043$

Sums	∓ 4324.5 + 3,23			
Log. 4324.5	3.63594		Log. 3.33	0.5224
Compl. 1. 22	8.65758		....	8.6576
Log. F=0,5544	9.74382		Log. f=0.043	8.6335
— 108″98	2.03734		+ 0″0065	7.8135
			— 108.98	

Reduction = — 108,97 = — 1′48″,97

The sign + of the first term serves only for circumpolar stars in their inferior passages.

TABLES
OF THE MEAN REFRACTIONS,
AND OF
THEIR CORRECTIONS
For the True Zenith Distances.

TABLE VI.
Of the Mean Refractions for the True Zenith Distances.

Dist. T.	Refrac.	*Dif.*	Dist. T.	Refrac.	*Dif.*	Dist. T.	Refrac.	*Dif.*
°	′ ″	″	°	′ ″	″	°	′ ″	″
0	0 0.0	1.0	25	0 26.4	1.2	50	1 7.4	2.4
1	1.0	1.0	26	27.6	1.3	51	1 9.8	2.6
2	2.0	1.0	27	28.9	1.2	52	1 12.4	2.6
3	3.0	1.0	28	30.1	1.3	53	1 15.0	2.8
4	4.0	1.0	29	31.4	1.3	54	1 17.8	2.9
5	5.0	1.0	30	32.7	1.3	55	1 20.7	3.1
6	6.0	1.0	31	34.0	1.4	56	1 23.8	3.2
7	7.0	1.0	32	35.4	1.4	57	1 27.0	3.4
8	8.0	1.0	33	36.8	1.4	58	1 30.4	3.6
9	9.0	1.0	34	38.2	1.5	59	1 34.0	3.8
10	10.0	1.0	35	39.7	1.5	60	1 37.8	4.0
11	11.0	1.0	36	41.2	1.5	61	1 41.8	4.3
12	12.0	1.1	37	42.7	1.6	62	1 46.1	4.6
13	13.1	1.0	38	44.3	1.6	63	1 50.7	4.9
14	14.1	1.1	39	45.9	1.6	64	1 55.6	5.2
15	15.2	1.0	40	47.5	1.7	65	2 0.8	5.7
16	16.2	1.1	41	49.2	1.8	66	2 6.5	6.1
17	17.3	1.1	42	51.0	1.8	67	2 12.6	6.5
18	18.4	1.1	43	52.8	1.9	68	2 19.1	7.3
19	19.5	1.1	44	54.7	1.9	69	2 26.4	7.9
20	20.6	1.1	45	56.6	2.0	70	2 34.3	8.7
21	21.7	1.2	46	58.6	2.1	71	2 43.0	9.5
22	22.9	1.1	47	1 0.7	2.1	72	2 52.5	10.6
23	24.0	1.2	48	1 2.8	2.3	73	3 3.1	11.9
24	25.2	1.2	49	1 5.1	2.3	74	3 15.0	13.2
25	26.4		50	1 7.4		75	3 28.2	

Continuation of TABLE VI.

Of the Mean Refractions for the True Zenith Distances.

Dist. T. °	′	Refrac. ′	″	Dif. ″
75		3	28.2	15.1
76		3	43.3	17.2
77		4	0.5	19.8
78		4	20.3	23.0
79		4	43.3	27.1
80	0	5	10.4	5.1
	10	5	15.5	5.2
	20	5	20.7	5.3
	30	5	26.0	5.5
	40	5	31.5	5.7
	50	5	37.2	5.9
81	0	5	43.1	6.0
	10	5	49.1	6.3
	20	5	55.4	6.5
	30	6	1.9	6.7
	40	6	8.6	6.9
	50	6	15.5	7.2
82	0	6	22.7	7.4
	10	6	30.1	7.7
	20	6	37.8	8.0
	30	6	45.8	8.3
	40	6	54.1	8.6
	50	7	2.7	9.0
83	0	7	11.7	9.3
	10	7	21.0	9.7
	20	7	30.7	10.1
	30	7	40.8	10.5
	40	7	51.3	11.0
	50	8	2.3	11.5
84	0	8	13.8	

Dist. T. °	′	Refrac. ′	″	Dif. ″
84	0	8	13.8	11.9
	10	8	25.7	12.5
	20	8	38.2	13.0
	30	8	51.2	13.7
	40	9	4.9	14.3
	50	9	19.2	14.9
85	0	9	34.1	15.7
	10	9	49.8	16.5
	20	10	6.3	17.3
	30	10	23.6	18.2
	40	10	41.8	19.2
	50	11	1.0	20.2
86	0	11	21.2	21.0
	10	11	42.2	22.5
	20	12	4.7	23.6
	30	12	28.3	24.9
	40	12	53.2	26.2
	50	13	19.4	27.9
87	0	13	47.3	29.4
	10	14	16.7	31.0
	20	14	47.7	32.9
	30	15	20.6	34.8
	40	15	55.4	37.3
	50	16	32.7	38.5
88	0	17	11.2	41.3
	10	17	52.6	43.6
	20	18	36.2	45.8
	30	19	22.0	49.1
	40	20	11.1	51.7
	50	21	2.8	54.5
89	0	21	57.3	

Dist. T. °	′	Refrac. ′	″	Dif ″
89	0	21	57.3	57.4
	10	22	54.7	60.6
	20	23	55.3	63.6
	30	24	58.9	66.9
	40	26	5.8	70.1
	50	27	15.9	73.5
90	0	28	29.4	76.8
	10	29	46.2	79.9
	20	31	6.1	83.2
	30	32	29.3	86.5
	40	33	55.8	89.6
	50	35	25.4	92.8
91	0	36	58.2	

TABLE VII.

Corrections for Refraction.

Barometer.			Fahrenheit's Thermometer.		
—	+	x	Deg.	y	*Dif.*
in. dec.	in. dec.				
29.85	29.85	0.0000	86	—0.0992	
29.8	29.9	0.0017	85	0.0967	25
29.7	30.0	0.0050	84	0.0942	25
29.6	30.1	0.0083	83	0.0917	25
29.5	30.2	0.0116	82	0.0892	25
29.4	30.3	0.0149	81	0.0866	26
29.3	30.4	0.0182	80	0.0840	26
29.2	30.5	0.0215	79	0.0814	26
29.1	30.6	0.0248	78	0.0788	26
29.0	30.7	0.0281	77	0.0762	26
28.9	30.8	0.0314	76	0.0736	26
28.8	30.9	0.0357	75	0.0710	26
28.7	31.0	0.0389	74	0.0684	26
28.6	31.1	0.0422	73	0.0657	27
28.5	31.2	0.0455	72	0.0630	27
28.4	31.3	0.0488	71	0.0603	27
28.3	31.4	0.0521	70	0.0576	27
28.2	31.5	0.0554	69	0.0549	27
28.1	31.6	0.0587	68	0.0521	28
28.0	31.7	0.0620	67	0.0493	28
27.9	31.8	0.0653	66	0.0465	28
27.8	31.9	0.0686	65	0.0437	28
27.7	32.0	0.0719	64	0.0409	28
27.6	32.1	0.0753	63	0.0381	28
27.5	32.2	0.0787	62	0.0353	28
27.4	32.3	0.0821	61	0.0325	28
27.3	32.4	0.0855	60	0.0296	29
27.2	32.5	0.0889	59	0.0267	29
27.1	32.6	0.0923	58	0.0238	29
27.0	32.7	0.0956	57	0.0209	29
			56	0.0180	29
			55	0.0150	30

USE OF THIS TABLE.

With the height of the Barometer, enter the first part of the Table, and take a number x, to which the sign — is to be prefixed if the height of the Barometer be less than 29.85 inches, but the sign + if it be greater than that number.

With the height of the mercury in Fahrenheit's thermometer, take a number y, in the second part, with the sign which precedes it.

The function of $(x+y+xy)$ will be the factor by which the mean refraction must be multiplied, in order to obtain the correction which it is necessary to apply to that refraction.

The product of xy may frequently be neg-

Continuation of TABLE VII.

Corrections for Refraction.

Fahrenheit's Thermometer.			Fahrenheit's Thermometer.		
Deg.	y	*Dif.*	Deg.	y	*Dif.*
54	−0.0120	30	29	+0.0685	35
53	0.0090	30	28	0.0720	35
52	0.0060	30	27	0.0755	35
51	0.0030	30	26	0.0791	36
50	0.0000	30	25	0.0827	36
49	+0.0030	30	24	0.0863	36
48	0.0061	31	23	0.0899	36
47	0.0092	31	22	0.0935	36
46	0.0123	31	21	0.0972	37
45	0.0155	32	20	0.1009	37
44	0.0187	32	19	0.1046	37
43	0.0219	32	18	0.1083	37
42	0.0251	32	17	0.1121	38
41	0.0283	32	16	0.1159	38
40	0.0315	32	15	0.1197	38
39	0.0347	32	14	0.1236	39
38	0.0380	33	13	0.1275	39
37	0.0413	33	12	0.1314	39
36	0.0446	33	11	0.1353	39
35	0.0480	34	10	0.1392	40
34	0.0514	34	9	0.1432	40
33	0.0548	34	8	0.1472	40
32	0.0582	34	7	0.1512	40
31	0.0616	34	6	0.1553	41
30	0.0650	34	5	0.1594	41

USE OF THIS TABLE.

lected; but it is necessary to pay attention to the algebraic signs of x and y in obtaining it.

PORTABLE
BAROMETRICAL TABLES,

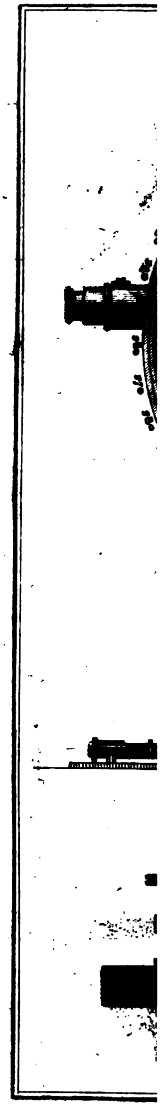
GIVING

The Differences of Level by a simple subtraction.

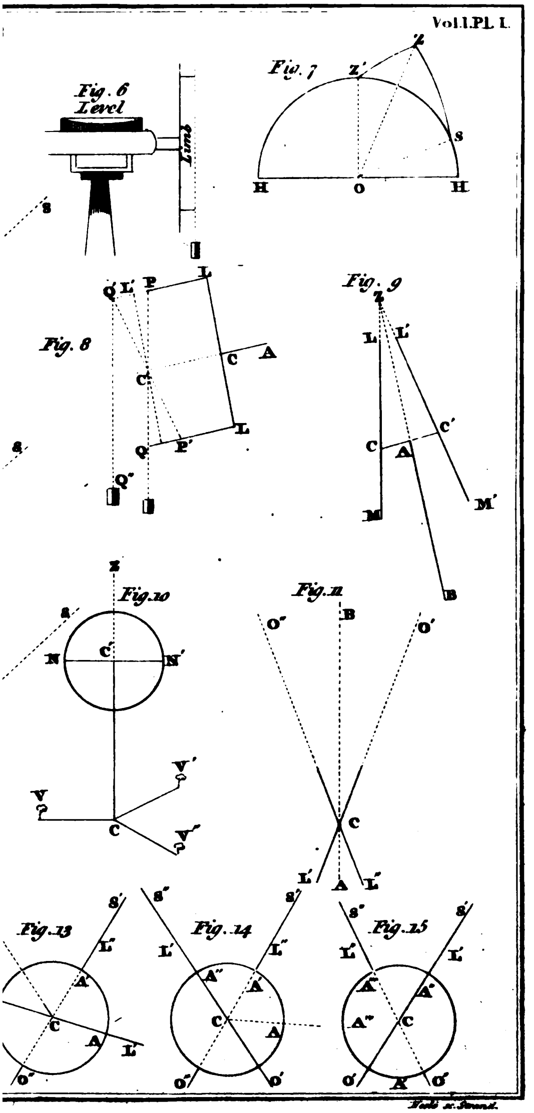
N. B. The number .76 of a metre, which the French assume for the mean pressure at the level of the sea, when expressed in decimals of an English yard, is .831136; but the first column of the Table only admitting of three figures, or thousandth parts, the last three figures of this number, *viz.* 136, have necessarily been left out in its calculation; hence the numbers in each column below .831 are all too little, and those above that number too great by a constant quantity, which is equal to the number in that column corresponding to .830 or .832, multiplyed by .136. Thus, for example; in the column answering to 48°, the required correction is $11.12 \times .136 = 1.50824$, which added to the number calculated from the Table, when the height of the barometer at the lower station does not exceed .831, will give the true difference of level sought. When this height of the barometer is greater than .831, the errors destroy each other, and the calculated heights do not require any correction.

The above calculation for the correction, however, may be avoided; for it will be sufficient to add 1.5 to all the calculated numbers that require correction, since the two extremes of this correction are $10.75 \times .136 = 1.462$, and $11.38 \times .136 = 1.54768$; half the sum of which is 1.50484. But as the first decimal will always be sufficient, the addition of 1.5 may be made mentally and without any trouble.

Translator.



Height of the Barometer.	*Sum of the Temperatures of the Air above the freezing Point, at the two extremities of the Column, or the values of T + t in degrees of Fahrenheit's Thermometer.*											Difference for 1°.
	42°	43°	44°	45°	46°	47°	48°	49°	50°	51°	52°	
yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	
835	43.92	43.96	44.00	44.05	44.09	44.14	44.19	44.23	44.28	44.33	44.38	.047
834	32.96	33.00	33.03	33.07	33.10	33.14	33.17	33.21	33.24	33.28	33.31	.035
833	21.99	22.01	22.04	22.06	22.08	22.11	22.13	22.15	22.47	22.20	22.22	.023
832	10.99	11.01	11.02	11.03	11.04	11.06	11.07	11.08	11.09	11.10	11.11	.012
831	0	0	0	0	0	0	0	0	0	0	0	.000
830	11.01	11.03	11.05	11.06	11.07	11.08	11.09	11.11	11.12	11.13	11.14	.012
829	22.02	22.05	22.07	22.12	22.14	22.16	22.19	22.21	22.24	22.26	22.28	.024
828	33.07	33.11	33.15	33.18	33.21	33.25	33.28	33.32	33.35	33.39	33.42	035
827	44.13	44.17	44.22	44.27	44.31	44.36	44.41	44.46	44.50	44.55	44.60	.047
826	55.19	55.25	55.31	55.37	55.42	55.48	55.54	55.60	55.66	55.72	55.78	.059
825	66.26	66.33	66.40	66.47	66.54	66.61	66.68	66.75	66.82	66.89	66.96	.070
824	77.34	77.42	77.51	77.58	77.67	77.75	77.83	77.92	78.00	78.08	78.16	.082
823	88.46	88.56	88.65	88.75	88.84	88.93	88.03	89.12	89.22	89.31	89.40	.094
822	99.56	99.67	99.78	99.88	99.99	100.10	100.20	100.31	100.41	100.52	100.63	.106
821	110.69	110.80	110.92	111.04	111.15	111.27	111.38	111.50	111.62	111.74	111.86	.117
820	121.82	121.95	122.08	122.21	122.34	122.46	122.59	122.72	122.85	122.98	123.11	.129
819	132.99	133.13	133.27	133.41	133.55	133.70	133.84	133.98	134.12	134.26	134.41	.141
818	144.17	144.33	144.48	144.63	144.78	144.94	145.09	145.24	145.40	145.55	145.70	.153
817	155.34	155.51	155.67	155.84	155.00	156.17	156.33	156.50	156.66	156.83	157.00	.165
816	166.55	166.73	166.90	167.08	167.26	167.45	167.62	167.79	167.97	168.15	168.33	.178
815	177.76	177.95	178.13	178.32	178.51	178.70	178.89	179.07	179.26	179.45	179.64	.188
814	188.97	189.17	189.37	189.57	189.77	189.97	190.17	190.37	190.57	190.77	190.99	.200
813	200.33	200.54	200.74	200.95	201.16	201.37	201.58	201.78	202.99	202.20	202.41	.208
812	211.46	211.69	211.91	212.14	212.36	212.58	212.82	213.04	213.27	213.49	213.72	.224
811	222.76	222.99	223.23	223.46	223.70	223.94	224.17	224.41	224.64	224.88	225.12	.236
810	234.04	234.29	234.53	234.78	235.03	235.28	235.53	235.77	236.02	236.27	236.52	.248
809	245.34	245.60	245.86	246.12	246.38	246.64	246.90	247.16	247.42	247.68	247.94	.260
808	256.72	256.99	257.27	257.54	257.81	258.08	258.35	258.63	258.90	259.17	259.44	.272
807	267.94	268.23	268.51	268.80	269.08	269.36	269.65	269.93	270.22	270.50	270.78	.284
806	279.29	279.58	279.88	280.17	280.47	280.77	281.06	281.36	281.65	281.95	282.25	296
805	289.98	290.28	290.59	290.90	291.20	291.51	291.82	292.13	292.43	292.74	293.05	.307
804	302.00	302.32	302.64	302.96	303.28	303.60	303.93	304.25	304.57	304.89	305.21	.321
803	313.39	313.72	314.06	314.39	314.72	315.06	315.39	315.72	316.05	316.39	316.72	.333
802	324.77	325.12	325.46	325.81	326.15	326.50	326.84	327.19	327.53	327.88	328.22	.345
801	336.20	336.55	336.91	337.27	337.62	337.98	338.34	338.70	339.05	339.41	339.77	.357
800	347.61	347.98	348.35	348.72	349.08	349.45	349.82	350.19	350.56	350.93	351.30	.369
799	359.04	359.42	359.80	360.18	360.56	360.95	361.33	361.71	362.09	362.47	362.85	.381
798	370.48	370.88	371.27	371.66	372.05	372.45	372.84	373.23	373.63	374.02	374.41	.393
797	381.95	382.36	382.76	383.17	383.57	383.98	384.38	384.79	385.19	385.60	386.00	.405
796	393.44	393.85	394.27	394.69	395.10	395.52	395.94	396.36	396.77	397.19	397.61	.417
795	404.90	405.33	405.77	406.20	406.63	407.06	407.49	407.93	408.35	408.79	409.22	.432
794	416.42	416.87	417.31	417.76	418.20	418.64	419.09	419.53	419.98	420.42	420.86	.444
793	427.94	428.40	428.85	429.31	429.76	430.22	430.67	431.13	431.58	432.04	432.50	.455
792	439.50	439.96	440.43	440.90	441.36	441.83	442.30	442.77	443.23	443.70	444.17	.467
791	451.02	451.50	451.99	452.47	452.95	453.43	453.91	454.40	454.88	455.36	455.84	.482
790	462.60	463.10	463.59	464.08	464.57	465.07	465.56	466.05	466.55	467.04	467.53	.493



Neele sc. Strand.

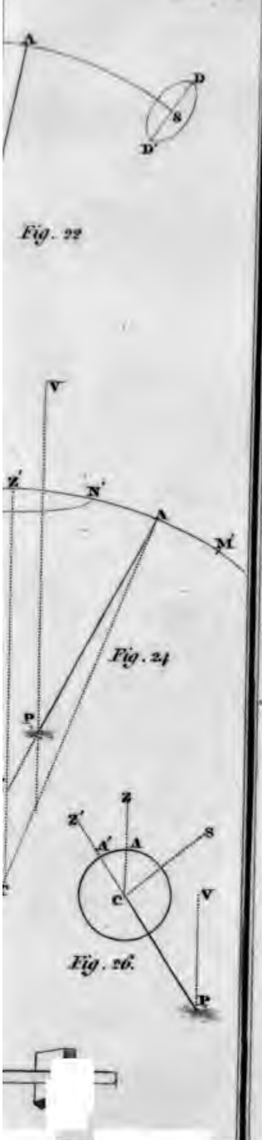
Sum of the Temperatures of the Air above the freezing Point at the two extremities of the Column, or the values of $T + t$ in degrees of Fahrenheit's Thermometer.

Height of the Barometer.	64°.	65°.	66°.	67°.	68°.	69°.	70°.	71°.	72°.	73°.	74°.	Difference for 1°.
yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	
835	44.93	44.97	45.02	45.07	45.11	45.16	45.20	45.25	45.30	45.34	45.39	.046
834	33.73	33.77	33.80	33.84	33.87	33.92	33.95	33.99	34.02	34.06	34.09	.035
833	22.50	22.52	22.54	22.57	22.59	22.61	22.63	22.66	22.68	22.70	22.73	.023
832	11.25	11.26	11.28	11.29	11.30	11.31	11.32	11.34	11.35	11.36	11.37	.012
831	0	0	0	0	0	0	0	0	0	0	0	.000
830	11.27	11.28	11.29	11.30	11.31	11.33	11.34	11.35	11.36	11.37	11.38	.011
829	22.54	22.57	22.59	22.62	22.64	22.66	22.69	22.71	22.74	22.76	22.78	.024
828	33.84	33.87	33.91	33.94	33.98	34.02	34.05	34.09	34.12	34.16	34.19	.036
827	45.16	45.20	45.25	45.29	45.34	45.39	45.43	45.48	45.52	45.57	45.62	.046
826	56.48	56.54	56.59	56.65	56.71	56.77	56.83	56.88	56.94	57.00	57.06	.058
825	67.80	67.87	67.94	68.01	68.08	68.16	68.23	68.30	68.37	68.44	68.51	.071
824	79.15	79.23	79.32	79.40	79.48	79.56	79.64	79.73	79.81	79.89	79.97	.082
823	90.51	90.61	90.70	90.80	90.89	90.98	91.08	91.17	91.27	91.36	91.45	.094
822	101.89	101.99	102.10	102.20	102.31	102.42	102.52	102.63	102.73	102.84	102.95	.106
821	113.27	113.39	113.51	113.63	113.75	113.87	113.99	114.11	114.23	114.35	114.46	.120
820	124.67	124.80	124.93	125.06	125.19	125.32	125.45	125.58	125.71	125.84	125.97	.130
819	136.11	136.25	136.39	136.53	136.67	136.81	136.95	137.09	137.23	137.51	137.51	.140
818	147.53	147.69	147.84	147.99	148.14	148.30	148.45	148.60	148.76	148.91	149.06	.153
817	158.97	159.14	159.30	159.47	159.63	159.80	159.96	160.13	160.29	160.46	160.63	.165
816	170.44	170.61	170.79	170.97	171.14	171.32	171.50	171.68	171.85	172.03	172.21	.177
815	181.92	182.11	182.29	182.48	182.67	182.86	183.05	183.23	183.42	183.61	183.80	.188
814	193.40	193.61	193.81	194.01	194.21	194.42	194.62	194.82	195.03	195.23	195.43	.203
813	204.91	205.12	205.32	205.53	205.74	205.95	206.16	206.36	206.57	206.78	206.99	.208
812	216.41	216.64	216.86	217.09	217.31	217.54	217.76	217.99	218.21	218.44	218.66	.225
811	227.93	228.17	228.41	228.65	228.89	229.14	229.38	229.62	229.86	230.10	230.34	.241
810	239.52	239.77	240.02	240.27	240.52	240.77	241.02	241.27	241.52	241.77	242.02	.250
809	251.06	251.32	251.58	251.84	252.10	252.36	252.62	252.88	253.14	253.40	253.66	.260
808	262.62	262.89	263.16	263.43	263.70	263.97	264.24	264.51	264.78	265.05	265.32	.270
807	274.24	274.52	274.81	275.09	275.38	275.67	275.95	276.24	276.52	276.81	277.10	.286
806	285.92	286.21	286.51	286.80	287.10	287.40	287.69	287.99	288.28	288.58	288.88	.296
805	296.74	297.05	297.35	297.66	297 97	298.28	298.69	298.99	299.30	299.61	299.92	.308
804	309.06	309.38	309.70	310.02	310.34	310.67	310.99	311.31	311.63	311.95	312.27	.321
803	320.73	321.06	321.40	321.73	322.06	322.39	322.72	323.06	323.39	323.72	324.05	.332
802	332.36	332.71	333.05	333.40	333.74	334.08	334.43	334.77	335.12	335.46	335.81	.344
801	344.09	344.45	344.81	345.17	345.53	345.89	346.25	346.61	346.97	347.33	347.69	.360
800	355.73	356.10	356.47	356.84	357.21	357.58	357.95	358.32	358.69	359.06	359.42	.370
799	367.43	367.81	368.19	368.57	368.95	369.34	369.72	370.10	370.48	370.86	371.24	.381
798	379.14	379.54	379.93	380.32	380.71	381.10	381.50	381.89	382.28	382.68	383.08	.393
797	390.88	391.29	391.69	392.10	392.50	392.91	393.31	393.72	394.12	394.53	394.94	.405
796	402.63	403.04	403.46	403.88	404.29	404.71	405.14	405.56	405.97	406.39	406.80	.417
795	414.40	414.83	415.26	415.69	416.12	416.55	416.98	417.41	417.84	418.91	418.69	.430
794	426.17	426.61	427.06	427.50	427.94	428.38	428.82	429.27	429.71	430.15	430.59	.442
793	437.95	438.41	438.86	439.32	439.77	440.23	440.68	441.14	441.59	442.05	442.50	.455
792	449.78	450.24	450.71	451.18	451.64	452.11	452.58	453.05	453.51	453.98	454.45	.467
791	461.59	462.07	462.55	463.03	463.50	463.98	464.46	464.94	465.42	465.90	466.38	.479
790	473.43	473.92	474.41	474.90	475.39	475.89	476.38	476.87	477.36	477.85	478.34	.491

Height of the Barometer.	*Sum of the Temperatures of the Air above the freezing Point at the two extremities of the Column, or the value of T+t in degrees of Fahrenheit's Thermometer.*											Difference for 1°
	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	30°	
yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.
789	463,13	463.63	464.14	464.64	465.14	465.64	466.15	466.65	467.15	467.66	468.16	.503
788	474.44	474.96	475.47	475.99	476.50	477.02	479.53	478.01	478.56	479.02	479.60	.515
787	485.68	486.21	486.74	487.27	487.79	488.32	488.85	489.38	489.90	490.43	490.96	.528
786	497.15	497.69	498.23	498.77	499.31	499.85	500.39	500.93	501.47	502.01	502.55	.540
785	508.51	509.06	509.62	510.17	510.72	511.27	511.82	512.38	512.93	513.49	514,04	.553
784	519.90	520.46	521.03	521.59	522.16	522.72	523.29	523.85	524.42	524.98	525.55	.565
783	531.30	531.88	532.46	533.03	533.61	534.19	534.77	535.35	535.92	556.50	537.08	.578
782	542.71	543.30	543.89	544.48	545.07	545.66	546.25	546.84	547.43	548.02	548.61	.590
781	554.14	554.74	555.34	555.95	556.55	557.15	557.76	558.36	558.96	559.56	560.16	.602
780	565.57	566,18	566.80	567.41	568.03	568.64	569.26	569.87	570.49	571.10	571.72	.615
779	577.03	577,66	578,28	578.91	579.54	580.16	580.79	581.42	582.05	582.67	583.30	.627
778	588.50	589,14	589.78	590.42	591.06	591.70	592.34	592.98	593.62	594.26	594.90	.640
777	600.00	600,65	601.30	601.96	602.61	603.26	603.91	604.56	605.22	605.87	606.52	.652
776	611.49	612.15	612,82	613.48	614.15	614.81	615.48	616.14	616.81	617.47	618.14	.665
775	623.00	623.68	624.35	625.03	625.71	626.38	627.06	627.74	628.42	629.09	629.77	.677
774	634.53	635.22	635,91	636.60	637.29	637.98	638.67	639.36	640·05	640.74	641.43	.690
773	646.07	646.77	647.48	648.18	648.88	649.58	650.29	650.99	651.69	652.40	653.10	.703
772	657.64	658.35	659.07	659.78	660.50	661.21	661.93	662.64	663.36	664.07	664.79	.715
771	669.21	669.94	670.66	671.39	672.12	672.84	673.57	674.30	675.03	675.75	676.48	.727
770	680,80	681.54	682.28	683.02	683.76	684.50	685.24	685.98	686.72	687.46	688.20	.740
769	692.41	693.16	693.91	694.66	695.42	696.17	696.92	697.67	698.43	699.18	699.93	.752
768	704.02	704.78	705.55	706.31	707.08	707.84	708.61	709.37	710.14	710.90	711.67	.765
767	715,66	716.44	717.21	717.99	718.77	719.54	720.32	721.10	721.88	722.65	723.43	.777
766	727.31	728,10	728.89	729.68	730.47	731.26	732.05	732.84	733.63	734.42	735.21	.790
765	738.97	739.77	740.58	741.38	742.19	742.99	743.79	744.50	745.40	746.21	747.01	.804
764	750.65	751.47	752,28	753.10	753.91	754.73	755.55	756.36	757.18	757.99	758.81	.816
763	762.35	763.18	764.01	764.84	765.67	766.49	767.32	768.15	768.98	769.81	770.64	.829
762	774.06	774.90	775.74	776.58	777.43	778.27	779.11	779.95	780.80	781.74	782.48	.842
761	785.79	786.64	787,50	788,35	789.21	790.06	790.91	791.77	792.62	793.47	794.33	.854
760	797.53	798.40	799,26	800.13	801.00	801.86	802.73	803.60	804.47	805.33	806.20	.867
759	809.30	810.18	811.06	811.94	812.82	813.69	814.57	815.45	816.33	817.21	818.09	.879
758	821.06	821.95	822.85	823,74	824.63	825.52	826.42	827.31	828.20	829.10	829.99	.892
757	832.85	833.75	834,66	835.56	836.47	837.37	838.28	839.18	840.09	840.99	841.90	.905
756	844.65	845,57	846.49	847.40	848.32	849.24	850.16	851.08	851.99	852.91	853.83	.918
755	856.47	857.40	858.33	859.26	860.19	861.12	862.06	862.99	863.92	864.85	865.78	.931
754	868.31	869.25	870.20	871.14	872.09	873.03	873.97	874.92	875.86	876.81	877.75	.944
753	880.16	881.12	882.07	883.03	883.99	884.94	885.90	886.86	887.82	888.77	889.73	.957
752	892.03	893.00	893.97	894.94	895.91	896.87	897.84	898.81	899.78	900.75	901.72	.969
751	903.91	904.89	905.88	906.86	907.84	908.82	909.81	910.79	911.77	912,76	913.74	.983
750	915.81	916.80	917.80	918.79	919.79	920.78	921.78	922.77	923.77	924.76	925.76	.995
749	927.72	928.73	929.74	930.75	931.76	932.77	933.78	934.79	935.80	936.81	937.82	1·01
748	939.65	940.67	941.69	942.71	943.73	944.75	945.77	946.79	947.81	948.83	949.85	1.02
747	951.60	952.63	953.66	954.69	955.72	956.75	957.78	958.81	959,84	960.87	961.90	1.03
746	963.56	964.61	965.66	966.71	967.76	968.81	969.86	970.91	971.96	973.01	974.06	1.05
745	975.54	976.70	977,76	978.82	979.88	980.94	982.00	983.06	984.12	985.18	986.24	1.06
744	987.53	988.60	989.67	990.74	991.81	992.88	993.95	995.02	996.09	997.16	998.23	1.07

Height of the Barometer.	*Sum of the Temperatures of the Air above the freezing Point at the two extremities of the Column, or the values of T+t in degrees of Fahrenheit's Thermometer.*										Difference for 1°
	31°	32°	33°	34°	35°	36°	37°	38°	39°	40°	41°	
	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	
789	468.66	469.17	469.67	470.17	470.67	471.18	471.68	472.18	472.69	473.19	473.69	.503
788	480.11	480.63	481.14	481.66	482.17	482.67	483.19	483.71	484.22	484.74	485.25	.515
787	491.49	492.02	492.55	493.08	493.61	494.14	494.67	495.20	495.73	496.26	496.79	.530
786	503.09	503.63	504.17	504.71	505.25	505.79	506.33	506.87	507.41	507.95	508.49	.540
785	514.59	515.15	515.70	516.25	516.80	517.35	517.91	518.46	519.01	519.56	520.12	.553
784	526.11	526.68	527.24	527.81	528.38	528.94	529.50	530.07	530.63	531.20	531.76	.565
783	537.66	538.23	538.81	539.39	539.96	540.54	541.11	541.69	542.26	542.84	543.42	.576
782	549.20	549.79	550.38	550.97	551.55	552.14	552.73	553.32	553.91	554.50	555.09	.589
781	560.76	561.36	561.97	562.57	563.17	563.77	564.37	564.98	565.58	566.18	566.78	.602
780	572.33	572.95	573.56	574.18	574.79	575.41	576.02	576.64	577.25	577.87	578.48	.615
779	583.93	584.55	585.18	585.81	586.43	587.06	587.69	588.32	588.94	589.57	590.20	.627
778	595.54	596.18	596.82	597.46	598.09	598.73	599.37	600.01	600.65	601.29	601.93	.639
777	607.17	607.83	608.48	609.13	609.78	610.44	611.09	611.74	612.30	613.05	613.70	.653
776	618.80	619.47	620.13	620.80	621.46	622.12	622.79	623.45	624.12	624.78	625.44	.664
775	630.45	631.12	631.80	632.48	633.15	633.83	634.51	635.19	635.86	636.54	637.22	.677
774	642.12	642.81	643.50	644.19	644.88	645.57	646.26	646.95	647.64	648.33	649.01	.690
773	653.80	654.50	655.21	655.91	656.61	657.31	658.01	658.72	659.42	660.12	660.82	.702
772	665.50	666.22	666.93	667.65	668.36	669.08	669.79	670.51	671.22	671.94	672.65	.715
771	677.21	677.93	678.66	679.39	680.11	680.84	681.57	682.30	683.02	683.75	684.48	.727
770	688.94	689.68	690.42	691.16	691.90	692.64	693.38	694.12	694.86	695.60	696.34	.740
769	700.68	701.44	702.19	702.94	703.69	704.45	705.20	705.95	706.71	707.46	708.21	.753
768	712.43	713.20	713.96	714.73	715.49	716.26	717.02	717.80	718.56	719.53	720.09	.765
767	724.21	724.98	725.76	726.54	727.32	728.10	728.88	729.65	730.43	731.21	731.99	.778
766	736.00	736.79	737.58	738.38	739.17	739.96	740.75	741.54	742.34	743.13	743.92	.791
765	747.81	748.62	749.42	750.22	751.03	751.83	752.63	753.43	754.24	755.04	755.84	.803
764	759.63	760.44	761.26	762.07	762.89	763.71	764.52	765.34	766.15	766.97	767.79	.816
763	771.47	772.30	773.12	773.95	174.78	775.61	776.44	777.26	778.09	778.92	779.75	.828
762	783.32	784.16	785.00	785.84	786.68	787.53	788.37	789.21	790.05	790.89	791.73	.841
761	795.18	796.04	796.89	797.75	798.60	799.45	800.31	801.16	802.02	802.87	803.72	.854
760	807.07	807.93	808.80	809.66	810.53	811.40	812.26	813.13	813.99	814.86	815.73	.866
759	818.97	819.85	820.73	821.61	822.49	823.37	824.25	825.13	826.01	826.89	827.77	.880
758	830.88	831.77	832.69	833.56	834.45	835.34	836.23	837.13	838.02	838.91	839.80	.892
757	842.80	843.71	844.61	845.52	846.42	847.33	848.23	849.14	850.04	850.95	851.86	.905
756	854.75	855.67	856.58	857.50	758.42	859.34	860.26	861.17	862.09	863.01	863.93	.918
755	866.71	867.64	868.57	869.50	870.43	871.37	872.30	873.23	874.16	875.09	876.02	.931
754	878.69	879.64	880.58	881.53	882.47	883.41	884.36	885.30	886.25	887.19	888.13	.944
753	890.69	891.64	892.60	893.55	894.51	895.47	896.42	897.38	898.33	899.29	900.25	.956
752	902.70	903.66	904.63	905.60	906.57	907.54	908.51	909.47	910.45	911.42	912.39	.970
751	914.72	915.70	916.69	917.67	918.65	919.63	920.61	921.60	922.58	923.56	924.54	.982
750	926.76	927.75	928.74	929.74	930.73	931.73	932.72	933.72	934.71	935.71	936.71	.995
749	938.83	939.84	940.85	941.86	942.87	943.88	944.89	945.90	946.91	947.92	948.89	1.01
748	950.87	951.89	952.91	953.93	954.95	955.97	956.99	958.01	959.03	960.05	961.07	1.02
747	962.94	963.98	965.02	966.06	967.10	968.14	969.18	970.22	971.26	972.30	973.34	1.04
746	975.11	976.16	977.21	978.26	979.31	980.36	981.41	982.46	983.51	984.56	985.61	1.05
745	987.30	988.36	989.42	990.48	991.54	992.60	993.66	994.72	995.78	996.84	997.90	1.06
744	999.30	1000.4	1001.4	1002.5	1003.6	1004.6	1005.7	1006.8	1007.9	1008.9	1010.0	1.07

Vol.I.Pl. 2



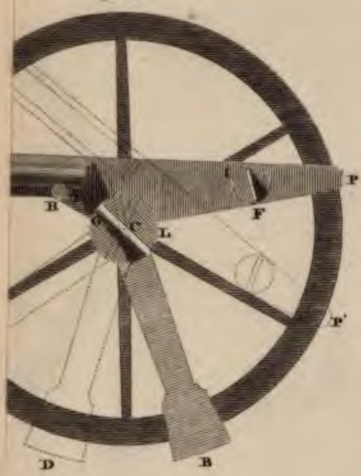
Sum of the Temperatures of the Air above the freezing Point at the two extremities of the Column, or the value of T+t in degrees of Fahrenheit's Thermometer.

53°	54°	55°	56°	57°	58°	59°	60°	61°	62°	63°	Difference for 1°
yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	
479.73	480.24	480.74	481.25	481.75	482.25	482.76	483.26	483.77	484.27	484.78	.504
491.48	492.00	492.51	493.03	493.54	494.06	494.57	495.09	495.60	496.12	496.63	.515
503.11	503.64	504.16	504.69	505.22	505.75	506.28	506.80	507.33	507.86	508.39	.528
514.98	515.55	516.06	516.60	517.14	517.68	518.22	518.93	519.30	519.84	520.38	.540
526.75	527.31	527.86	528.41	528.96	529.52	530.07	530.62	531.18	531.73	532.28	.553
538.54	539.11	539.67	540.24	540.80	541.37	541.93	542.50	543.06	543.68	544.20	.565
550.36	550.93	551.57	552.09	552.66	553.24	553.82	554.40	554.97	555.55	556.13	.577
562.17	562.76	563.35	563.94	564.53	565.12	565.71	566.30	566.89	567.48	568.07	.590
574.02	574.62	575.23	575.83	576.43	577.03	577.63	578.24	578.84	579.44	580.04	.602
585.85	586.47	587.08	587.70	588.31	588.93	589.54	590.16	590.77	591.39	592.01	.615
597.73	598.35	598.98	599.61	600.23	600.86	601.49	602.12	602.74	603.37	604.00	.627
609.61	610.25	610.89	611.53	612.16	612.80	613.44	614.08	614.72	615.36	616.00	.639
621.51	622.17	622.82	623.47	624.12	624.78	625.43	626.08	626.74	627.39	628.04	.653
633.42	634.09	634.75	635.42	636.08	636.75	637.41	638.08	638.74	639.41	640.07	.665
645.35	646.02	646.70	647.38	648.05	648.73	649.41	650.09	650.78	651.44	652.12	.677
657.29	657.98	658.67	659.36	660.05	660.74	661.43	662.12	662.81	663.50	664.19	.690
669.25	669.95	670.66	671.36	672.06	672.76	673.46	674.17	674.87	675.57	676.27	.702
681.22	681.94	682.65	683.37	684.08	684.80	685.51	686.23	686.94	687.66	688.37	.715
693.22	693.94	694.67	695.40	696.12	696.85	697.58	698.31	699.03	699.76	700.49	.727
705.22	705.96	706.70	707.44	708.18	708.92	709.66	710.40	711.14	711.88	712.62	.740
717.23	717.99	718.74	719.49	720.24	721.00	721.75	722.50	723.26	724.01	724.76	.753
729.27	730.04	730.80	731.57	732.33	733.10	733.86	734.63	735.39	736.16	736.93	.765
741.32	742.11	742.88	743.66	744.44	745.22	746.00	746.77	747.55	748.33	749.10	.778
753.41	754.20	754.99	755.78	756.56	757.35	758.14	758.93	759.72	760.51	761.30	.789
765.48	766.29	767.09	767.89	768.69	769.50	770.30	771.10	771.91	772.71	773.51	.803
777.58	778.38	779.20	780.01	780.83	781.65	782.46	783.28	784.09	784.91	785.74	.816
789.70	790.53	791.35	792.18	793.01	793.84	794.67	795.49	796.32	797.15	797.95	.828
801.82	802.66	803.51	804.35	805.19	806.03	806.87	807.72	808.56	809.40	810.24	.842
813.97	814.83	815.68	816.54	817.39	818.24	819.10	819.95	820.81	821.66	822.51	.854
826.14	827.00	827.87	828.74	829.60	830.47	831.34	832.21	833.07	833.94	834.81	.867
838.33	839.21	840.09	840.97	841.84	842.72	743.60	844.48	845.36	846.24	847.12	.879
850.51	851.41	852.30	853.19	854.08	854.98	855.87	856.76	857.66	858.55	859.44	.893
862.72	863.63	864.53	865.44	866.34	867.25	868.16	869.06	869.96	870.87	871.78	.905
874.95	875.87	876.78	877.70	878.62	879.54	880.46	881.37	882.29	883.21	884.13	.918
887.19	888.12	889.05	889.98	890.91	891.85	892.78	893.71	894.64	895.57	896.50	.931
899.45	900.40	901.34	902.29	903.23	904.17	905.12	906.06	907.01	907.91	908.89	.944
911.73	912.68	913.64	914.59	915.55	916.51	917.46	918.42	919.37	920.33	921.29	.956
924.02	924.99	925.96	926.93	927.90	928.87	929.84	930.81	931.78	932.75	933.72	.970
936.33	937.32	938.30	939.28	940.26	941.25	942.23	943.21	944.20	945.18	946.16	.983
948.65	949.65	950.64	951.64	952.63	953.63	954.62	955.62	956.61	957.61	958.61	.995
961.01	962.02	963.03	964.04	965.05	966.06	967.07	968.08	969.08	970.09	971.10	1.01
973.32	974.35	975.38	976.41	977.44	978.47	979.50	980.53	981.56	982.58	983.61	1.03
985.71	986.75	987.79	988.83	989.87	990.91	991.95	992.99	994.03	995.07	996.10	1.04
998.10	999.15	1000.2	1001.2	1002.3	1003.3	1004.4	1005.4	1006.5	1007.5	1008.6	1.05
1010.6	1011.6	1012.7	1013.7	1014.8	1015.9	1016.9	1018.0	1019.0	1020.1	1021.2	1.06
1022.9	1024.0	1025.0	1026.1	1027.2	1028.3	1029.4	1030.4	1031.5	1032.6	1033.7	1.08

Vol.I.Pl. 3

Fig. 30

Fig. 31



Neele sc. Strand

Vol.I.Pl.4

38

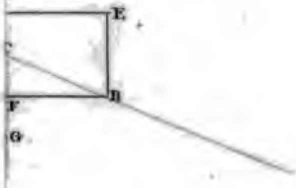


Fig. 42

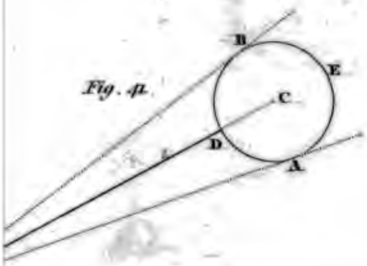
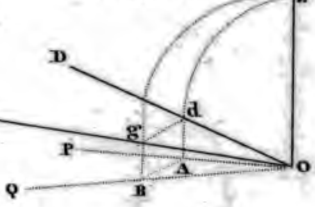


Fig. 44



Neele sc. Strand.

Sum of the Temperatures of the Air above the freezing Point at the two extremities of the Column, or the values of $T + t$ in degrees of Fahrenheit's Thermometer.

Height of the Barometer.	42°	43°	44°	45°	46°	47°	48°	49°	50°	51°	52°	Difference for 1°.
yds.	yds.	yds.	yds.	yds.	yds.	yds	yds.	yds.	yds.	yds.	yds.	yds.
743	1023.7	1024.8	1025.8	1026.9	1028.0	1029.1	1030.2	1031.2	1032.3	1033.4	1034.5	1.10
742	1035.8	1036.9	1038.0	1039.1	1040.2	1041.3	1042.4	1043.5	1044.6	1045.7	1046.8	1.10
741	1048.1	1049.2	1050.4	1051.5	1052.6	1053.7	1054.8	1056.0	1057.1	1058.2	1059.3	1.12
740	1060.4	1061.6	1062.7	1063.8	1064.9	1066.1	1067.2	1068.3	1069.5	1070.6	1071.7	1.13
739	1072.7	1073.9	1075.0	1076.2	1077.3	1078.4	1079.6	1080.7	1081.9	1083.0	1084.2	1.14
738	1085.1	1086.3	1087.4	1088.6	1089.7	1090.9	1092.0	1093.2	1094.3	1095.5	1096.7	1.15
737	1097.6	1098.7	1099.9	1101.0	1102.2	1103.4	1104.5	1105.7	1106.8	1108.0	1109.2	1.16
736	1110.1	1111.2	1112.4	1113.6	1114.7	1115.9	1117.1	1118.3	1119.4	1120.6	1121.8	1.17
735	1122.4	1123.6	1124.8	1126.0	1127.1	1128.3	1129.5	1130.7	1131.9	1133.1	1134.3	1.19
734	1134.9	1136.1	1137.3	1138.5	1139.7	1140.9	1142.1	1143.3	1144.5	1145.7	1146.9	1.20
733	1147.3	1148.5	1149.8	1151.0	1152.2	1153.4	1154.6	1155.9	1157.1	1158.3	1159.5	1.22
732	1159.8	1161.1	1162.3	1163.5	1164.7	1166.0	1167.2	1168.4	1169.7	1170.9	1172.1	1.23
731	1172.3	1173.6	1174.8	1176.1	1177.3	1178.5	1179.8	1181.0	1182.3	1183.5	1184.7	1.24
730	1184.8	1186.1	1187.3	1188.6	1189.8	1191.1	1192.3	1193.6	1194.8	1196.1	1197.4	1.25
729	1197.4	1198.6	1199.9	1201.2	1202.4	1203.7	1205.0	1206.3	1207.5	1208.8	1210.1	1.27
728	1209.9	1211.2	1212.5	1213.8	1215.0	1216.3	1217.6	1218.9	1220.2	1221.5	1222.8	1.29
727	1222.5	1223.8	1225.1	1226.4	1227.7	1229.0	1230.3	1231.6	1232.9	1234.2	1235.5	1.30
726	1235.1	1236.4	1237.7	1239.0	1240.3	1241.7	1243.0	1244.3	1245.6	1246.9	1248.2	1.31
725	1247.6	1249.0	1250.3	1251.6	1252.9	1254.3	1255.6	1256.9	1258.3	1259.6	1260.9	1.33
724	1260.3	1261.7	1263.0	1264.4	1265.7	1267.0	1268.4	1269.7	1271.1	1272.4	1273.7	1.34
723	1272.9	1274.3	1275.6	1277.0	1278.3	1279.7	1281.0	1282.3	1283.6	1285.0	1286.4	1.35
722	1285.6	1286.9	1288.3	1289.6	1291.0	1292.4	1293.7	1295.1	1296.4	1297.8	1299.2	1.36
721	1298.3	1299.6	1301.0	1302.4	1303.7	1305.1	1306.5	1307.9	1309.2	1310.6	1312.0	1.37
720	1311.0	1312.4	1313.8	1315.2	1316.5	1317.9	1319.3	1320.7	1322.1	1323.5	1324.9	1.39
719	1323.7	1325.1	1326.5	1327.9	1329.3	1330.7	1332.1	1333.5	1334.9	1336.3	1337.7	1.40
718	1336.4	1337.8	1339.3	1340.7	1342.1	1343.5	1344.9	1346.4	1347.8	1349.2	1350.6	1.42
717	1349.1	1350.6	1352.0	1353.5	1354.9	1356.3	1357.8	1359.2	1360.7	1362.1	1363.5	1.44
716	1361.9	1363.4	1364.8	1366.3	1367.7	1369.2	1370.6	1372.1	1373.5	1375.0	1376.4	1.45
715	1374.7	1376.1	1377.6	1379.0	1380.5	1382.0	1383.4	1384.9	1386.3	1387.8	1389.3	1.46
714	1387.5	1388.9	1390.4	1391.9	1393.3	1394.8	1396.3	1397.8	1399.2	1400.7	1402.2	1.47
713	1400.3	1401.8	1403.3	1404.8	1406.2	1407.7	1409.2	1410.7	1412.2	1413.7	1415.2	1.49
712	1413.1	1414.6	1416.1	1417.6	1419.1	1420.6	1422.1	1423.6	1425.1	1426.6	1428.1	1.50
711	1425.9	1427.4	1429.0	1430.5	1432.0	1433.5	1435.0	1436.6	1438.1	1439.6	1441.1	1.52
710	1438.8	1440.4	1441.9	1443.4	1444.9	1446.5	1448.0	1449.5	1451.1	1452.6	1454.1	1.53
709	1451.7	1453.3	1454.8	1456.4	1457.9	1459.4	1461.0	1462.5	1464.1	1465.6	1467.1	1.54
708	1464.6	1466.2	1467.7	1469.3	1470.8	1472.4	1473.9	1475.5	1477.0	1478.6	1480.2	1.55
707	1477.6	1479.1	1480.7	1482.3	1483.8	1485.4	1487.0	1488.6	1490.1	1491.7	1493.3	1.57
706	1490.4	1492.0	1493.5	1495.1	1496.7	1498.3	1499.9	1501.4	1403.0	1504.6	1506.2	1.58
705	1503.4	1505.0	1506.6	1508.2	1509.8	1511.4	1513.0	1514.6	1516.2	1517.8	1519.4	1.60
704	1516.4	1518.0	1519.6	1521.2	1522.8	1524.5	1526.1	1527.7	1529.4	1530.9	1532.5	1.61
703	1529.4	1531.1	1532.7	1534.3	1535.9	1537.6	1538.2	1540.8	1542.5	1544.1	1545.7	1.63
702	1542.4	1544.1	1545.7	1547.4	1549.0	1550.6	1552.3	1553.9	1555.6	1557.2	1558.8	1.64
701	1555.4	1557.1	1558.7	1560.4	1562.0	1563.7	1565.3	1567.0	1568.6	1570.3	1571.9	1.65
700	1568.6	1570.2	1571.9	1573.5	1575.2	1576.9	1578.5	1580.2	1581.8	1583.5	1585.1	1.66
699	1581.6	1583.2	1584.9	1586.6	1588.2	1589.9	1591.6	1593.3	1594.9	1596.6	1598.3	1.67
698	1594.7	1596.4	1598.1	1599.8	1601.4	1603.1	1604.8	1606.5	1608.2	1609.9	1611.6	1.69
697	1607.8	1609.5	1611.2	1612.9	1614.6	1616.3	1618.0	1619.7	1621.4	1623.1	1624.8	1.70

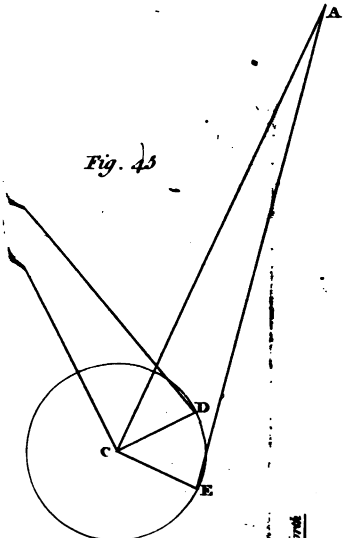


Fig. 45

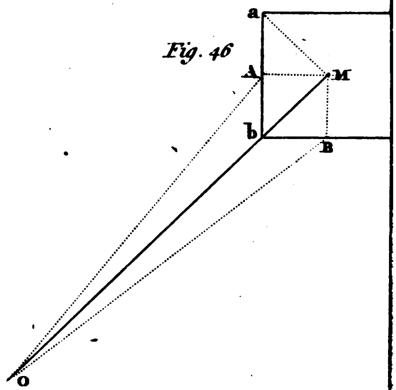


Fig. 46

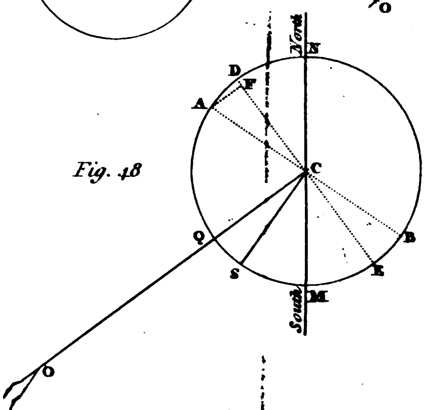


Fig. 48

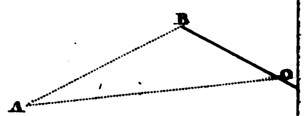


Fig. 49

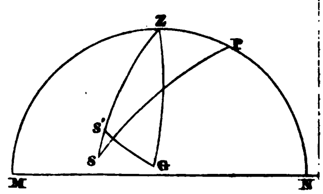


Fig. 52

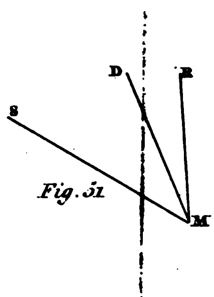


Fig. 51

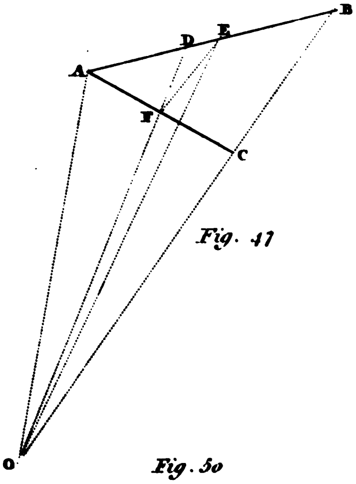
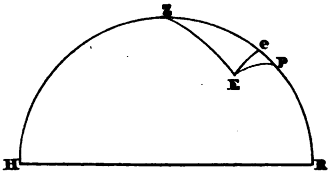
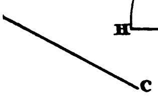


Fig. 47

Fig. 50





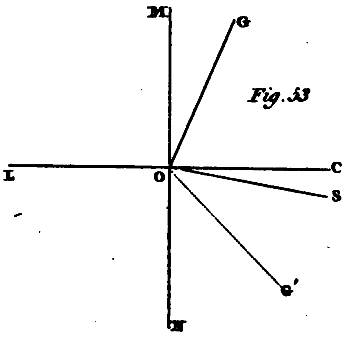


Fig. 53

Sum of the Temperatures of the Air above the freezing Point, at the two extremities of the Column, or the value of T + *t in degrees of Fahrenheit's Thermometer.*

Height of the Barometer.	64°	65°	66°	67°	68°	69°	70°	71°	72°	73°	74°	Difference for 1°.
yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.	yds.
696	1658.8	1660.5	1662.3	1664.0	1665.7	1667.4	1669.1	1670.9	1672.6	1674.3	1676.0	1.72
695	1672.2	1673.9	1675.7	1677.4	1679.1	1680.9	1682.6	1684.3	1686.1	1687.8	1689.6	1.73
694	1685.7	1687.5	1689.2	1691.0	1692.7	1694.4	1696.3	1698.0	1699.8	1701.5	1703.2	1.74
693	1699.2	1701.0	1702.7	1704.5	1706.2	1708.0	1709.7	1711.5	1713.2	1715.0	1716.8	1.75
692	1712.7	1714.4	1716.2	1718.0	1719.7	1721.5	1723.3	1725.1	1726.8	1728.6	1730.4	1.77
691	1726.2	1728.0	1729.8	1731.6	1733.4	1735.1	1736.9	1738.7	1740.5	1742.3	1744.1	1.79
690	1739.8	1741.6	1743.4	1745.2	1747.0	1748.8	1750.6	1752.4	1754.2	1756.0	1757.8	1.80
689	1753.3	1755.1	1757.0	1758.8	1760.6	1762.4	1764.2	1766.1	1767.9	1769.7	1771.5	1.82
688	1766.9	1768.8	1780.6	1772.4	1774.2	1776.1	1777.9	1779.7	1781.6	1783.4	1785.2	1.83
687	1780.5	1782.4	1784.2	1786.1	1787.9	1789.7	1791.6	1793.4	1795.3	1797.1	1799.0	1.84
686	1794.2	1796.0	1797.9	1809.7	1801.6	1803.5	1805.3	1807.2	1809.0	1810.9	1812.8	1.86
685	1807.8	1809.7	1811.5	1813.4	1815.3	1817.2	1819.1	1820.9	1822.8	1824.7	1826.6	1.88
684	1821.5	1823.4	1825.3	1827.2	1829.0	1830.9	1832.8	1834.7	1836.6	1838.5	1840.4	1.89
683	1835.2	1837.1	1839.0	1840.9	1842.8	1844.7	1846.6	1848.5	1850.4	1852.3	1854.2	1.90
682	1848.9	1850.8	1852.7	1854.7	1856.6	1858.5	1860.4	1862.3	1864.3	1866.2	1868.1	1.92
681	1862.6	1864.6	1866.5	1868.4	1870.3	1872.3	1874.2	1876.1	1878.1	1880.0	1882.0	1.93
680	1876.3	1878.3	1880.2	1882.2	1884.1	1886.1	1888.0	1890.0	1891.9	1893.9	1895.8	1.95
679	1890.2	1892.1	1894.1	1896.0	1898.0	1899.9	1901.9	1903.9	1905.8	1907.8	1909.8	1.96
678	1904.0	1905.9	1907.9	1909.9	1911.8	1913.8	1915.8	1917.8	1919.7	1921.7	1923.7	1.97
677	1917.7	1919.7	1921.7	1923.7	1925.6	1927.6	1929.6	1931.6	1933.6	1935.6	1937.6	1.99
676	1931.6	1933.6	1935.6	1937.6	1939.6	1941.6	1943.6	1945.6	1947.6	1949.6	1951.6	2.00
675	1945.4	1947.4	1949.5	1951.5	1953.5	1955.5	1957.5	1959.6	1961.6	1963.6	1965.6	2.02
674	1959.3	1961.4	1963.4	1965.4	1967.4	1969.5	1971.5	1973.5	1975.6	1977.6	1979.6	2.02
673	1973.2	1975.3	1977.3	1979.4	1981.4	1983.4	1985.5	1987.5	1989.6	1991.6	1993.7	2.04
672	1987.1	1989.2	1991.2	1993.3	1995.3	1997.4	1999.4	2001.5	2003.5	2005.6	2007.7	[illegible]
671	2001.1	2003.1	2005.2	2007.3	2009.3	2011.4	2013.5	2015.6	2017.6	2020.7	2021.8	[illegible]
670	2015.0	2017.1	2019.2	2021.3	2023.3	2025.4	2027.5	2029.6	2031.7	2033.8	2035.9	[illegible]
669	2029.0	2031.1	2033.2	2035.3	2037.4	2039.5	2041.6	2043.7	2045.8	2047.9	2050.0	[illegible]
668	2042.9	2045.0	2047.2	2049.3	2051.4	2053.5	2055.6	2057.8	2059.9	2062.0	2064.1	[illegible]
667	2056.9	2059.1	2061.2	2063.4	2065.5	2067.6	2069.8	2071.9	2074.1	2076.2	2078.3	[illegible]
666	2071.0	2073.2	2075.3	2077.5	2079.6	2081.8	2083.9	2086.1	2088.2	2090.4	2092.5	[illegible]
665	2085.1	2087.2	2089.4	2091.5	2093.7	2095.9	2098.0	2100.2	2102.3	2104.5	2106.7	[illegible]
664	2099.2	2101.3	2103.5	2105.7	2107.8	2110.0	2112.2	2114.4	2116.5	2118.7	2120.9	[illegible]
663	2113.3	2145.5	2117.7	2119.9	2122.0	2124.2	2126.4	2128.6	2130.8	2133.0	2135.2	[illegible]
662	2127.4	2129.6	2131.8	2134.0	2136.2	2138.5	2140.7	2142.9	2145.1	2147.3	2149.5	[illegible]
661	2141.5	2143.7	2146.0	2148.2	2150.4	2152.7	2154.9	2157.1	2159.3	2161.6	2163.8	[illegible]
660	2155.7	2158.0	2160.2	2162.5	2164.7	2166.9	2169.2	2171.4	2173.7	2175.9	2178.1	[illegible]
659	2169.9	2172.2	2174.4	2176.7	2178.9	2181.2	2183.4	2185.7	2187.9	2190.2	2192.4	[illegible]
658	2184.1	2186.3	2188.6	2190.8	2193.1	2195.4	2197.6	2199.8	2202.1	2204.4	2206.8	[illegible]
657	2198.3	2200.6	2202.8	2205.1	2207.4	2209.7	2212.0	2214.2	2216.5	2218.8	2221.1	[illegible]
656	2212.6	2214.9	2217.2	2219.5	2221.8	2224.1	2226.4	2228.7	2231.0	2233.3	2235.6	[illegible]
655	2226.8	2229.1	2231.5	2233.8	2236.1	2238.4	2240.7	2243.1	2245.4	2247.7	2250.	[illegible]
654	2241.1	2243.5	2245.8	2248.1	2250.5	2252.8	2255.1	2257.4	2259.8	2262.1	2264.	[illegible]
653	2255.5	2257.9	2260.2	2262.6	2264.9	2267.2	2269.6	2271.9	2274.3	2276.6	2278.	[illegible]
652	2269.8	2272.2	2274.5	2276.9	2279.2	2281.6	2283.9	2286.3	2288.6	2291.0	2293.	[illegible]
651	2284.2	2286.5	228 .9	2291.3	2293.6	2296.0	2298.4	2300.8	2303.1	2305.5	2307.	[illegible]
650	2298.6	2301.0	2303.4	2305.8	2308.1	2310.5	2312.9	2315.3	2317.7	2318.1	2322.	[illegible]

A Table of the Correction for Latitude.

LATITUDE.	CORRECTION.
0°	+ $\frac{1}{352}$ of the calculated height.
5	+ $\frac{1}{358}$
10	+ $\frac{1}{375}$
15	+ $\frac{1}{407}$
20	+ $\frac{1}{460}$
25	+ $\frac{1}{548}$
30	+ $\frac{1}{705}$
35	+ $\frac{1}{1030}$
40	+ $\frac{2}{2030}$
45	0
50	− $\frac{1}{2030}$
55	− $\frac{1}{1030}$
60	− $\frac{1}{705}$
65	− $\frac{1}{548}$
70	− $\frac{1}{460}$
75	− $\frac{1}{407}$
80	− $\frac{1}{375}$
85	− $\frac{1}{358}$
90	− $\frac{1}{352}$

From the equator to the latitude of 45°, the correction must be added to the calculated difference of level, but it should be subtracted from it, from the latitude of 45° to the pole.

N.B. The height of the mercury in the barometer, inserted in the first columns of each page of the preceding Barometrical Table, being given in decimals of an English yard, it is necessary, of course, when this height is observed in inches and decimals of an inch, first to divide by 36, the number which expresses it; the Table is then to be entered with the quotient. Suppose, for example, that the height of the Barometer was observed to be 29.7 inches; then, $\frac{29.7}{36} = \frac{4.95}{6} = .825$, the number with which the Table is to be entered.

END OF

Fig 56

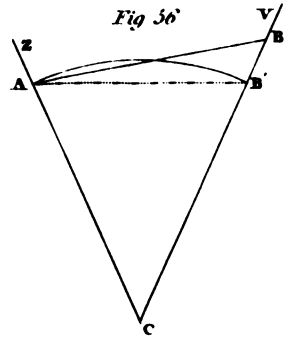


Fig 58

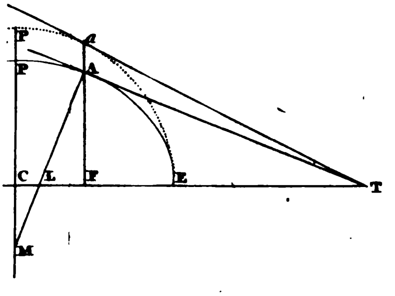
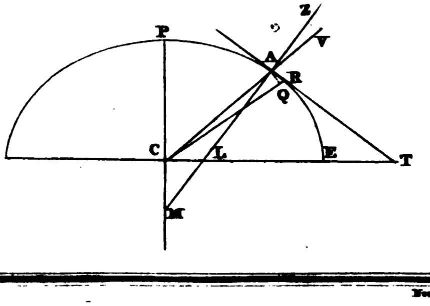


Fig. 61



Neele sc. Strand.